Programme & Book of Abstracts

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Programme

Friday, 21st August

08:00–08:50 *Morning reception and registration*

08:50–09:00 *Opening address*

09:00–10:00 *Invited talk (session chair: Nina Gierasimczuk)* **Magdalena Ortiz**, *It's All Connected: Logic and Knowledge Representation for Graph Data*

10:00–10:25 *Coffee break*

10:25–12:30 *Contributed talks (session chair: Fredrik Engström)*

- 10:25 **Hans van Ditmarsch, Valentin Goranko and Yanjing Wang**, *The Quest for Ultimate Ignorance I. Iterated ignorance is complex!*
- 10:50 **Alexander V. Gheorghiu**, *A Proof-theoretic Foundation for Mathematics*
- 11:15 **Barteld Kooi and Allard Tamminga**, *From Tonk to Harmony via Identity and Cut*
- 11:40 **Karl Nygren**, *Announcing alternatives*
- 12:05 **Mina Young Pedersen and Sonja Smets**, *Logical Modeling of Belief Polarization*

12:30–13:45 *Lunch*

13:45–15:00 *Contributed talks (session chair: Graham Leigh)*

- 13:45 **Michał Walicki**, *The definability of truth in LSO*
- 14:10 **Luca Castaldo**, *Determinateness in a Classical Theory of Nonclassical Truth*
- 14:35 **Charles Crumpler**, *Cut for Negative Free Classical Core Logic with Equality and a Variable-binding, Term-forming Operator*

15:00–15:25 *Coffee break*

15:25–16:15 *Contributed talks (session chair: Santiago Jockwich)*

- 15:25 **Søren Brinck Knudstorp**, *Undecidability in Relevant Logic*
- 15:50 **Davide Botticchio**, *A Non-bivalent Naïve Validity Theory*

Saturday, 22nd August

08:00–09:00 *Morning buffet*

09:00–10:00 *Invited talk (session chair: Valentin Goranko)* **Carsten Schürmann**, *Towards a Logic of Zero-Knowledge* (joint work with Markus Krabbe Larsen and Abhi Shelat)

10:00–10:25 *Coffee break*

10:25–12:30 *Contributed talks (session chair: Juha Kontinen)*

- 10:25 **Wessel Kroon**, *Partition-Based Topic-Sensitive Intentional Modals*

10:50 **Orvar Lorimer-Olsson**, *Semantic Plurivalence is Logically Idempotent*
11:15 **Santiago Jockwich Martinez**, *Should we worry about Duplication?*
11:40 **Uwe Wolter**, *Licensing Hypothetical Reasoning*
12:05 **Qianli Zeng**, *A Jónsson-Tarski-Goldblatt Representation and Frame Definability for Noncontingency Logic*

12:30–13:45 *Lunch*

13:45–14:45 *Invited talk (session chair: Mina Young Pedersen)* **Vera Koponen**, *Logical convergence laws from the perspective of statistical relational artificial intelligence*

14:45–15:25 *Coffee break*

15:25–16.25 ***Panel discussion***

16.25–17:10 ***SLS General Meeting***

20:30 *Conference Dinner*

Sunday, 23rd August

08:00–09:30 *Morning buffet*

09:30–10:30 *Invited talk (session chair: Patrick Blackburn)* **Salvatore Florio**, *Sets and Pluralities*

10:30–10:55 *Coffee break*

10:55–12:35 *Contributed talks (session chair: Søren Brinck Knudstorp)*

10:55 **Sama Agahi**, *Self-Sufficiency of Logics: A Framework for Metalogical Constraints*
11:20 **Frederik J. Andersen**, *On the epistemic significance of convergence in logical theorizing*
11:45 **Ludovica Conti**, *Weak Impredicativity and Frege's* Grundgesetze
12:10 **Alexandru Petrişor**, *Truthmaker Semantics and Hempelian Explanation in Mathematics*

12:35–13:45 *Lunch*

13:45–15:00 *Contributed talks (session chair: Alessandro Bruni)*

13:45 **Mirko Engler** (joint work with Benjamin Zayton), *Rigid Theories*
14:10 **Sebastian Enqvist-Pyk**, *From Herbrand schemes to functional interpretation*
14:35 **Lars Kristiansen**, *From a computability-theoretic point of view: The real numbers and their representations*

15:00–15:20 *Coffee break*

15:20–16:10 *Contributed talks (session chair: Anders Schlichtkrull)*

15:20 **Chirine Yasmine Laghjichi**, *Atomic steps of computation and β-reductions*
15:45 **Jørgen Villadsen**, *Formalizing a Concise Axiomatic System for Classical Propositional Logic in Isabelle/HOL*

16:10–16:20 *Closing remarks & farewell*

Invited talks

Sun, 09:30

Sets and Pluralities

Salvatore Florio
University of Oslo

There is a close but puzzling connection between sets and pluralities. While a set is fully characterized by the plurality of its elements, the set-theoretic paradoxes seem to show that not all pluralities correspond to sets. I examine and compare two ways in which theories of pluralities have been used to explain sets. The first, famously associated with Cantor, takes a set to be one object formed from a plurality. The second, famously advocated by Russell, interprets set talk as plurality talk in disguise. I show that Cantor's ideas lead to a simple, consistent, and powerful theory of set abstraction.

Logical convergence laws from the perspective of statistical relational artificial intelligence

Sat, 13:45

Vera Koponen
Uppsala University

The topic of logical convergence laws in finite model theory has a more than 50 year long history. I will survey more recent results on convergence laws which consider probability distributions on finite structures, and formal logics, that are motivated by questions in statistical relational artificial intelligence, a branch of artifical intelligence that combines the logic oriented and probability oriented schools of artificial intelligence.

Fri, 09:00

It's All Connected: Logic and Knowledge Representation for Graph Data

Magdalena Ortiz
TU Wien

Decidable logics have been fundamental to the development of correct and reliable data-centric intelligent systems. In particular, Description Logics (DLs) are a powerful and well-understood toolbox of decidable logics for describing and reasoning about graphs. As AI permeates more and more of our interactions with information and knowledge, reliable reasoning techniques that enable us to reach sound and verifiable conclusions are more important than ever. In this talk, we will discuss some emerging challenges for reasoning with graph data, a cornerstone of modern information management. The dominant graph-data paradigms have undergone major standardisation efforts in recent years, reopening old questions and raising new ones. The emergence of GQL and SQL/PGQ as standard query languages for property graphs resolves enduring obstacles to logic-mediated graph querying, while the rise of new constraint languages for graphs such as SHACL reveals a rich landscape of novel reasoning problems, including validation, static analysis, and constraint learning. These are intimately connected to description logics but demand revisiting basic assumptions about open- and closed-world reasoning, fixed-point semantics, and inconsistency tolerance. Both threads point to the same conclusion: some of the most exciting questions about graph data in intelligent systems are, at their core, problems of decidable logic, reasoning, and knowledge representation.

Towards a Logic of Zero-Knowledge

Sat, 09:00

Carsten Schürmann
IT University of Copenhagen

In cryptography, a Σ-protocol is a 3-round cryptographic protocol satisfies several important properties, including two special-soundness, completeness, and honest verifier zero-knowledge properties. These protocols are widely-used in larger cryptographic constructions and form the basis for a logic to reason about zero-knowledge.

The goal of my talk is threefold. First, I will try to give a gentle introduction into Σ-protocols and what the associated properties mean. Second, I will introduce a handful of composition (reasoning) rules to construct new protocols from existing ones, and finally, I demonstrate that this logic of zero-knowledge can handle complex examples from the literature including signatures schemes that can be used for age verification. All claims presented in this talk were successfully and efficiently formalized in the Rocq proof assistant.

This talk is based on joint work with Markus Krabbe Larsen and abhi shelat.

Contributed talks

Sun, 10:55

Self-Sufficiency of Logics: A Framework for Metalogical Constraints

Sama Agahi
Stockholm University

Can a logic do its own metatheory? Must it be able to? The first is a technical question about which logical resources are required for various metatheoretic tasks; the second is a question in the philosophy of logic about the correctness criteria of logics. The technical question has a long pedigree, going back at least to Gödel's observation that his completeness proof for classical first-order logic relies on the Law of Excluded Middle. Later work by Gödel, Kreisel [7], and McCarty [9] established the canonical case of what I call *self-sufficiency failure*: the unprovability of intuitionistic completeness with respect to standard model-theoretic semantics. Carter [2, 3, 4] systematised the question for intermediate logics and counter-model completeness, and similar questions have repeatedly been raised across the non-classical landscape: see e.g. [1, 10, 11, 12, 13]. The literature is rich but fragmented.

In this talk I present a unified framework, organised around what I call *self-sufficiency of a logic with respect to a metatheoretic property*: $\mathcal{L}$ is self-sufficient with respect to ϕ if establishing $\phi(\mathcal{L})$ requires no *logical* resources beyond those $\mathcal{L}$ itself provides. With Δ the non-logical background theory of the metatheory and $\mathcal{L}^M$ the metalogical version of $\mathcal{L}$, basic self-sufficiency is given by the schema

$$\Delta \vDash_{\mathcal{L}^M} \phi(\mathcal{L}).$$

This says that, relative to Δ, the logical resources of $\mathcal{L}$ suffice to establish $\phi(\mathcal{L})$. The schema can be varied along three dimensions—the target logic $\mathcal{L}$, the property ϕ, and the non-logical background Δ—and each of the three components does substantive work.

Take the property ϕ first. The notation $\phi(\mathcal{L})$ does not denote the application of a fixed predicate to $\mathcal{L}$ as an argument; rather, $\phi(\mathcal{L})$ abbreviates a formula in the language of the metatheory expressing that $\mathcal{L}$ has the property ϕ, with $\mathcal{L}$ entering as a parameter—through whichever of its consequence relation, proof system, or semantics the property concerns. Counter-model completeness, for example, is rendered as $\phi(\mathcal{L}) := \forall \Gamma \left(\Gamma \nvdash_{\mathcal{L}} \bot \ \rightarrow \ \exists M \, (M \vDash \Gamma) \right)$. A closed formula whose content is fixed once $\mathcal{L}$ is. This flexibility lets the framework span the metatheoretic tasks at issue—completeness, soundness, decidability, definability—which divide into *definitional* tasks such as giving syntax and semantics and *inferential* tasks such as proving $\phi(\mathcal{L})$.

The non-logical background Δ—typically a fragment of set theory or arithmetic—supplies the structures needed for syntax, semantics, and the expression of ϕ. It cannot be left implicit, because non-logical resources can interact with logical ones in ways that obscure the very logical differences a self-sufficiency claim is meant to track. A well-known example is provided by the Diaconescu–Goodman–Myhill theorem [5, 6], which shows that the Axiom of Choice, added to a constructive set theory, yields the equivalent of the Law of Excluded Middle. I call this phenomenon metalogical flattening: where the background already supplies the inferential power of LEM, the question whether a constructive logic is self-sufficient

for some property has been flattened out of existence. A meaningful question of *logical* self-sufficiency therefore requires that Δ be carefully chosen and the choice clearly signalled.

Since the schema characterises a metalevel investigation, the target logic must be available at the metalevel. $\mathcal{L}^M$ is a higher-order extension of $\mathcal{L}$ furnishing the resources to quantify over the syntactic and semantic structures ϕ refers to while adding no new logical validities. $\mathcal{L}$ is *lifted* to its metalogical version. For intermediate logics Carter's Meta-operator does this work and is shown to be injective in the propositional case, so we really are using the same logic at the metalevel. Carter's results cannot be assumed to carry over to the wider landscape the framework is meant to cover.

When $\mathcal{L}$ fails to be self-sufficient with respect to ϕ, a natural further question is which additional resources would suffice. Inspired by Makinson's [8] technique for bridging monotonic and non-monotonic logics—which extends a logic by an *assumption*, a *rule*, or a *restriction* on admissible models—the framework's *modular* variant records explicitly a resource R used in establishing $\phi(\mathcal{L})$:

$$\Delta \left|\frac{R}{\mathcal{L}^M}\right. \phi(\mathcal{L}).$$

What the framework adds is a clean separation: R may be added to $\mathcal{L}^M$ as logical strength or to Δ as non-logical background—a distinction that matters for whether $\mathcal{L}$ has been recaptured or merely extended.

Several further variants can be formulated. A *familial* version relativises to a class of logics, requiring only that some member supply the needed resources. A *syntactic* variant replaces semantic consequence with provability, and a *decidability* variant asks whether $\mathcal{L}$ can settle $\phi(\mathcal{L})$ either way. A *reversal* variant, inspired by reverse mathematics, asks which resources are required rather than merely sufficient.

A central purpose of the framework is to make existing discussions of metalogical constraints both mutually comparable and more fine-grained. An example of the latter use is given by Tanaka and Girard's notion of *paraconsistent ascent*—the demand that a normatively motivated paraconsistent logic keep Explosion out of its metatheory. It is ambiguous between basic self-sufficiency, where the same paraconsistent $\mathcal{L}$ must suffice, and the familial variant, where some non-explosive logic must. Their argument establishes only the latter, while their conclusion is stated for the former; the framework makes the gap visible. Casting such discussions in a common schema brings into focus distinctions sometimes obscured—between target logic and metalogic, and between using a logical resource and admitting it as logical—and so opens space for sharpening the arguments themselves.

The schema also makes natural a technical question that can be pursued within it: the structure of the *self-sufficiency classes* $[\![\phi]\!]_\Delta = \{\, \mathcal{L} : \Delta \vDash_{\mathcal{L}^M} \phi(\mathcal{L}) \,\}$ as a measure of the metalogical demandingness of properties.

[1] Guillermo Badia, Zach Weber, and Patrick Girard. *Paraconsistent metatheory: New proofs with old tools.* ***Journal of Philosophical Logic***, 51(4):825–856, 2022.

[2] Nathan C. Carter. *Reflexive Intermediate Logics.* PhD thesis, Indiana University, 2004.
[3] Nathan C. Carter. *Reflexive intermediate propositional logics.* ***Notre Dame Journal of Formal Logic***, 47(1):39–62, 2006.
[4] Nathan C. Carter. *Reflexive intermediate first-order logics.* ***Notre Dame Journal of Formal Logic***, 49(1):75–95, 2008.
[5] Radu Diaconescu. *Axiom of choice and complementation.* ***Proceedings of the American Mathematical Society***, 51(1):176–178, 1975.
[6] Nicolas Goodman and John Myhill. *Choice implies excluded middle.* ***Zeitschrift fur mathematische Logik und Grundlagen der Mathematik***, 24(25–30), 1978.
[7] Georg Kreisel. *On weak completeness of intuitionistic predicate logic.* ***The Journal of Symbolic Logic***, 27(2):139–158, 1962.
[8] David Makinson. *Bridges from Classical to Nonmonotonic Logic.* Texts in Computing Vol. 5. King's College, 2005.
[9] David C. McCarty. *Incompleteness in intuitionistic metamathematics.* ***Notre Dame Journal of Formal Logic***, 32(3):323–358, 1991.
[10] Koji Tanaka and Patrick Girard. *Against classical paraconsistent metatheory.* ***Analysis***, page anac093, 2023.
[11] Andrew Tedder. *Nonclassical logicians can use a classical metatheory without abandoning their principles.* ***Analysis***, 2025.
[12] Neil Tennant. *Relevance in reasoning.* Pages 696–725, 2005.
[13] Timothy Williamson. *Logic, metalogic and neutrality.* ***Erkenntnis***, 79:211–231, 2014.

On the Epistemic Significance of Convergence in Logical Theorizing

Sun, 11:20

Frederik J. Andersen
University of Copenhagen

This paper presents some reflections on the interesting—and yet underexplored—epistemological flip side of logical disagreement[1], viz., the epistemic significance of agreement (or convergence) in logic. When two incompatible logical theories have a non-empty intersection including the content expressed by a given entailment-claim, we intuitively want this to increase our confidence in the overlap. We should be more confident about the content expressed by the entailment-claim since it is common ground between the theories. Or should we? While the idea of convergence is familiar from the philosophy of science, there is almost no discussion of the nature and epistemic significance of convergence in logical theorizing to be found in the philosophical literature. Perhaps convergence in logic is epistemically significant because converging logical theories constitute independent methods of reasoning confirming the same results, e.g., that Modus Ponens is a deductively valid inference, and when independent methods confirm the same results, we have better reason to be confident in said results. While this idea might seem initially appealing, there are serious challenges to it upon reflection. The first challenge lies in understanding what it means for methods of reasoning to be independent of each other on a generic level and explaining why convergence of such independent methods is *epistemically* significant. A second challenge is deciding how (or if at all) logical theories may be considered independent methods of reasoning, the convergence of which is epistemically significant. As will become clear, even if we can say something useful about the first of these challenges, the second is not a straightforward matter.

[1] Anandi Hattiangadi. *Logical disagreement.* In Conor McHugh, editor, ***Metaepistemology***, pages 88–106. Oxford University Press, 2018.

[2] Ole Thomassen Hjortland. *Disagreement about logic.* ***Inquiry***, 65(6):660–682, 2022.

[1] The topic *logical disagreement* has received some attention in recent literature. See for instance [1, 2].

Fri, 15:50

A Non-bivalent Naïve Validity Theory

Davide Botticchio
Universitat de Barcelona

In this paper, by adding a validity predicate Val to a propositional language, we develop a validity theory that fully and correctly captures its metalinguistic validity. More precisely, for any inference, metainference, and higher-order metainference, they are valid (invalid) iff the object language Val-sentences that express their validity (invalidity) are true in every model. We precisely formalize these *desiderata* with the so-called V-schemata. For example, for inferences, an inference is valid ($\Gamma \vDash \Delta$) iff the Val-sentence that expresses this fact is true in every model ($\vDash Val(\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner)$), and an inference is invalid ($\Gamma \nvDash \Delta$) iff the negated Val-sentence that expresses this fact is true in every model ($\vDash \neg Val(\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner)$). We generalize this schemata for higher level n-metainferences, for every n. We say that a validity theory is *naïve* iff it satisfies all the V-schemata. This project faces two challenges. First, weak validity and logical principles are inconsistent because of the well-known Validity Paradox (Beall and Murzi [3]). Second, all existing validity theories, even those that avoid the paradox, get some features of validity wrong. More precisely, we show that, in all validity theories based on a logic with a bivalent validity (i.e. in which inferences and n-metainferences are either valid or invalid, and not both), either the V-schema for valid inferences or V-schema for invalid inferences must fail. To overcome this limitation, we build a validity theory based on *non-bivalent* logics, logics in which inferences and n-metainferences can be both valid and invalid, or neither of the two (Pailos [7], et al. [1]). We provide independent justification for the plausibility of these logics by generalizing the bilateralist framework developed by Ripley [8]. We adopt the bilateralist idea that which inferences are valid is to be explained in terms of conditions on assertion and denial of sentences, where denial is not defined as the complement of assertion, but as a primitive notion that must be understood independently of assertion. We extend this framework to inferences as well: denial of an inference (invalidity) is to be understood and established independently of the negation of its assertion (non-validity). We define a logic, which we call $L_{st/ts}^{\nvDash \vDash / \vDash \nvDash}$, that has some valid and invalid inferences and n-metainferences, for every n. We show that $L_{st/ts}^{\nvDash \vDash / \vDash \nvDash}$ has same validities as the logic ST_ω, and so the same validities as classical logic (Barrio et al. [2], Pailos [5]), and the same invalidities as TS_ω, and so no non-invalidities in the language without propositional constants (Pailos [6]). We build a fixed-point interpretation Φ_{Val} for the validity predicate taking inspiration from Meadows [4]. We define the validity theory $LV_{st/ts}^{\nvDash \vDash / \vDash \nvDash}$ by taking the Strong Kleene valuations v_{Val} of the fixed point Φ_{Val} and evaluating inferences and n-metainferences according to the logic $L_{st/ts}^{\nvDash \vDash / \vDash \nvDash}$. We prove that this non-bivalent validity theory is consistent and naïve: it avoids paradoxes and satisfies all the V-schemata. We argue that $L_{st/ts}^{\nvDash \vDash / \vDash \nvDash}$ shares some features with other non-classical logics. It is analogous to ST_ω in its capability to deal with paradoxes while maintaining classical validities. The relation between its validities and invalidities mirrors the relation between truths and falsities in LP. Similarly to ST_ω, the intuitive explanation for why the theory

avoids the derivation of the validity paradox is that its validities are not closed under the rules they assert to be valid. However, unlike ST_ω, the non-bivalence of the theory provides an intuitive explanation of this feature of the logic.

[1] Eduardo Barrio, Camillo Fiore, and Federico Pailos. *Non-bivalent validity.* ***Studia Logica***, pages 1–35, 2026.

[2] Eduardo Alejandro Barrio, Federico Pailos, and Damian Szmuc. *A hierarchy of classical and paraconsistent logics.* ***Journal of Philosophical Logic***, 49(1):93–120, 2020.

[3] Jeffrey Charles Beall and Julien Murzi. *Two flavors of Curry's paradox.* ***The Journal of Philosophy***, 110(3):143–165, 2013.

[4] Toby Meadows. *Fixed points for consequence relations.* ***Logique et Analyse***, 227:333–357, 2014.

[5] Federico Pailos. *A fully classical truth theory characterized by substructural means.* ***The Review of Symbolic Logic***, 13(2):249–268, 2020.

[6] Federico Pailos. *Empty logics.* ***Journal of Philosophical Logic***, 51(6):1387–1415, 2022.

[7] Federico Pailos. *Naïve non-substructural solutions to the validity paradox.* ***Synthese***, 206(1):39, 2025.

[8] David Ripley. *Paradoxes and failures of cut.* ***Australasian Journal of Philosophy***, 91(1):139–164, 2013.

Fri, 14:10

Determinateness in a Classical Theory of Nonclassical Truth

Luca Castaldo
Ludwig-Maximilians-Universität München

Axiomatic theories of truth are severely constrained by the Liar and related semantic paradoxes. As is well known, no consistent theory formulated in classical logic can contain a truth predicate that is at once type-free, fully disquotational, and fully compositional. A consistent theory of truth must therefore restrict at least one of three things: the language, the underlying logic, or the basic truth-theoretic principles (cf. [3]).

A natural response to this is to distinguish between paradoxical and non-paradoxical sentences. For the latter, the thought goes, the restrictions forced on us by the semantic paradoxes are no longer necessary: non-paradoxical sentences can obey the principles that fail in general. In this sense, they are *admissible*. This talk presents a new axiomatic theory of truth and admissibility.

There is a long-standing tradition of axiomatic approaches to truth and admissibility. One idea in this tradition is that admissibility can help contain the impact of the semantic paradoxes: Even if a theory cannot validate certain principles in full generality, it may still recover them for the admissible sentences. For instance, suppose that a theory S restricts the interaction between truth and negation, so that the principle $\neg \mathrm{T}\varphi \leftrightarrow \mathrm{T}(\neg\varphi)$ does not hold unrestrictedly. Then S may be strengthened by adding the axiom (where $\mathrm{F}\varphi :\leftrightarrow \mathrm{T}(\neg\varphi)$):

$$\forall\varphi\,(A\varphi \rightarrow (\neg\mathrm{T}\varphi \leftrightarrow \mathrm{F}\varphi))\,.$$

Admissibility has often been treated as a definable notion, as for example in [12, 13], [5, 3, 4], and more recently in [1]. In recent years, however, an alternative approach has emerged. Field [7] and Fujimoto and Halbach [6] take admissibility—under the labels of *strong classicality* and *determinateness*, respectively—to be a genuinely primitive notion, not definable in terms of truth and falsity alone.[1]

Field's theory, INT, extends the Strong Kleene theory of transparent truth PKF—Partial Kripke-Feferman [10]—with a strong classicality predicate S. Since not every instance $\varphi \vee \neg\varphi$ of excluded middle is available in INT, S is used to recover the safe instances via

$$\emptyset \Rightarrow \forall\varphi\,(\mathrm{S}\varphi \rightarrow (\varphi \vee \neg\varphi))\,.$$

By contrast, Fujimoto and Halbach's theories, CD and CD^+, add determinateness to a "thoroughly classical" ([6], p. 220) theory of truth. In these theories, formulated over classical logic, truth commutes with all connectives and quantifiers, but full disquotation for truth itself does not hold unrestrictedly; determinateness is then used to recover its safe instances via

$$\forall\varphi\,(\mathrm{D}\varphi \rightarrow (\mathrm{TT}\varphi \leftrightarrow \mathrm{T}\varphi))\,.$$

Despite pulling in opposite directions—INT is a *nonclassical* theory of *nonclassical* truth, whereas CD and CD^+ are *classical* theories of *classical* truth—both theories may be viewed as embodying Kripke's notion of *groundedness* from [11]. And

[1] Whether determinateness needs to be primitive within Fujimoto and Halbach's theory is somewhat controversial, as argued in [1].

they both show that axiomatizing admissibility together with truth yields systems which are conceptually rich and mathematically strong.

The present talk continues this line of investigation, suggesting a third approach: an axiomatization of determinateness over a *classical* theory of *nonclassical* truth, i.e., a theory that preserves classical logic, but treats truth itself in a nonclassical way. The theory, KFD—for Kripke-Feferman with determinateness—will be shown to have a number of attractive features and to be significantly stronger (proof-theoretically) than its extant alternatives.

From a philosophical perspective, the "most fundamental decision for the truth theorist is whether to sacrifice classical logic for transparency [of truth] or transparency for classical logic" ([6], p. 259). Unlike INT, which sacrifices classical logic, KFD sacrifices transparency, like CD^+. I will not enter this discussion in the talk. Rather, I will compare KFD to the alternative CD^+. The crucial difference between them is that KFD retains the disquotational principle $\forall\varphi(\mathrm{T}\mathrm{T}\varphi \leftrightarrow \mathrm{T}\varphi)$, while CD^+ retains the completeness axiom $\forall\varphi(\neg\mathrm{T}\varphi \rightarrow \mathrm{F}\varphi)$ instead. I will argue, by appeal to *blind-deduction* examples (see [8, 9], [6], [1]), that disquotation is more central than completeness relative to both the ordinary and the theoretical role of truth.

From a technical perspective, KFD is shown to be proof-theoretically equivalent to the theory $\widehat{\mathsf{ID}}_2$ of two-times iterated fixed-points (see e.g. [2]). This goes far beyond the proof-theoretic strength of CD^+, which in [6] has been shown to be arithmetically equivalent to the system $\mathsf{RA}_{<\varepsilon_{\varepsilon_0}}$ of ramified analysis up to any ordinal $\alpha < \varepsilon_{\varepsilon_0}$. Time permitting, I will also discuss the implications for Field's INT. Field shows that INT is at least as strong as $\mathsf{RA}_{<\varepsilon_0}$ and conjectures that this lower bound is sharp. However, by a straightforward adaptation of the lower-bound argument for KFD, INT can be shown to be proof-theoretically stronger than $\mathsf{RA}_{<\varepsilon_0}$: more precisely, INT is equivalent to the sub-theory of KFD in which the induction schema is replaced by the so-called internal induction.

[1] Luca Castaldo and Carlo Nicolai. *On classical determinate truth.* ***The Review of Symbolic Logic***, pages 1–27, 2025.

[2] Solomon Feferman. *Iterated inductive fixed-point theories: Application to Hancock's conjecture.* In G. Metakides, editor, ***The Patras Symposion***, volume 109, pages 171–196. Elsevier, 1982.

[3] Solomon Feferman. *Toward useful type-free theories. I.* ***The Journal of Symbolic Logic***, 49(1):75–111, 1984.

[4] Solomon Feferman. *Reflecting on incompleteness.* ***The Journal of Symbolic Logic***, 56(1):1–49, 1991.

[5] Solomon Feferman. *Axioms for determinateness and truth.* ***The Review of Symbolic Logic***, 1(2):204–217, 2008.

[6] Kentaro Fujimoto and Volker Halbach. *Classical determinate truth I.* ***The Journal of Symbolic Logic***, 89(1):218–261, 2024.

[7] Hartry Field. *The power of naive truth.* ***The Review of Symbolic Logic***, 15(1):225–258, 2022.

[8] Kentaro Fujimoto. *Deflationism beyond arithmetic.* ***Synthese***, 196(3):1045–1069, 2019.

[9] Kentaro Fujimoto. *The function of truth and the conservativeness argument.* ***Mind***, 131(521):129–157, 2022.

[10] Volker Halbach and Leon Horsten. *Axiomatizing Kripke's theory of truth.* ***The Journal of Symbolic Logic***, 71(2):677–712, 2006.
[11] Saul Kripke. *Outline of a theory of truth.* ***The Journal of Philosophy***, 72(19):690–716, 1975.
[12] William Reinhardt. *Remarks on significance and meaningful applicability.* ***Mathematical Logic and Formal Systems***, 94:227, 1985.
[13] William Reinhardt. *Some remarks on extending and interpreting theories with a partial predicate for truth.* ***Journal of Philosophical Logic***, 15(2):219–251, 1986.

Weak Impredicativity and Frege's *Grundgesetze*

Sun, 11:45

Ludovica Conti
University of Vienna

The debate on impredicativity shows that this term can refer to different phenomena ([11]). In this talk, I will examine various forms of what Gödel termed "weak impredicativities" ([**?**]) and consider whether they could inform a consistent revision of Frege's logicist project ([15]).

At least three notions of predicativity/impredicativity are usually distinguished. These are "strict" predicativity, "relative or classical" predicativity, and "generalised or constructive" predicativity. Firstly, "strict" predicativity ([7]) is a syntactic notion captured by the following characterisation: a definition is said to be impredicative if its *definiens* contains a quantifier that binds a variable of the same order as the *definiendum*; in this case, the *definiendum* is one of the possible substitutional instances of the quantified *definiens* and apparently blocks the evaluation of the latter.[1] This notion of predicativity is the most austere, requiring the rejection of large portions of mathematics. In particular, with respect to this criterion, the induction scheme is impredicative; therefore, only fragments of arithmetic can be admitted ([10]). Secondly, "classical" predicativity ([10], [13] and [**?**]) is also known as "predicativity relative to natural numbers". This concept arose from Feferman and Kreisel's parallel investigation into the limits of predicativity, i.e. examining the extent to which predicative theories could be developed once natural numbers were accepted as given (despite their strictly impredicative definition). This interpretation of predicativity appears to be more promising, at least from a mathematical point of view. In this approach, natural numbers are assumed to be given, and a ramified hierarchy of second-order arithmetical theories with ramified comprehension is provided. Each stage is represented by a theory indexed by an ordinal available in the previous stage, and the limit of predicativity is identified as the first ordinal that cannot be proved predicatively (which has been proven to be Γ_0). Thirdly, "generalised" ([4], [18], [22]) or "constructive" predicativity ([5], [3], [1]) is instead a more recent criterion, introduced in order to capture a different shade of predicativity as constructivity. In virtue of this criterion we can classify as predicative also many inductive definitions.

With respect to this widely accepted taxonomy, I would argue that the notion of predicativity involved in strict and classical predicativism is originally syntactic. Conversely, the notion of generalised or constructive predicativity stems from an ontological characterisation of the structure of predicative entities. This characterisation is very similar to that captured by the set-theoretic criterion of well-foundedness (cf. [5], [3]). I support the idea that the weakly predicative phenomena that Gödel had in mind involve this ontological notion of predicativity, and I propose referring to the others as forms of "strong" impredicativity.

The second part of the talk is devoted to analysing second-order impredicativity in Frege's *Grundgesetze*. This investigation is guided by the idea that the predicat-

[1] A semantical description is maybe more usual. A definition is said to be "impredicative" if it contains an unrestricted quantifier that binds a variable of the same order of the *definiendum* – then, the *definiendum* is a substitutional instance of the bound variable.

ive subsystems that have been proposed so far – [16], [23], [14], [2] – are obtained by applying "strongly" predicative restrictions, which in this context mean "strict" predicativity.[2] I will briefly discuss these results to demonstrate the interesting parallels with those of strict predicativism, as mentioned above. Strongly predicative restrictions of *Grundgesetze* can indeed derive a fragment of second-order Peano arithmetic (PA^2) equivalent to Robinson arithmetic (Q). In this case, too, predicativism requires us to renounce induction, even though it is not assumed as an axiom, but rather derived as a theorem. Furthermore, the derivation of arithmetic does not proceed, not even partially, via Frege's definitions, as most of the original definitions, particularly the crucial notion of "ancestral" (cf. note 3), are precluded. Therefore, even with respect to the Neologicist proposal (cf. [12]), a predicativist restriction would undermine the derivation of Frege's theorem (FT), i.e. the derivation of second-order Peano axioms from second-order logic augmented with Hume's principle and Frege's definitions (ancestral, weak ancestral, predecessor and natural number).[3] In other words, strong predicativism excludes a feature that appears to be involved in the derivation of Russell's paradox, but was also necessary for the logicist derivation of Frege's theorem.

Conversely, I emphasise that the effect of a "weakly predicative" approach has so far been unexplored, and I aim to demonstrate that it would introduce an interesting modification to Frege's *Grundgesetze* because it would prevent the specification of Russell's concept ($Rx \leftrightarrow \exists X(x = \epsilon X \wedge \neg Xx)$), yet remain open to Frege's vocabulary particularly the definitions of ancestral, weak ancestral, predecessor and natural number (cf. note 3). Furthermore, it would enable the derivation of second-order Peano axioms, including induction.

In the final part of the talk, we will examine a potentially weaker form of impredicativity that remains compatible with both Gödel suggestion and Frege's logicist purposes. More precisely, we will identify and test a consistent and powerful predicative restriction of the comprehension axiom schema: $\forall\alpha\mathcal{R}(D,\alpha) \leftrightarrow \forall y(\phi(y) \rightarrow \mathcal{R}(\alpha, y))$ – where $\mathcal{R}$ is the characteristic relation for elements of the type of the *definiendum* (D), i.e. the relation used to provide its identity conditions. We will demonstrate that such a scheme is compatible with the Fregean vocabulary and the weakest form of impredicativity discussed so far. This will enable us to derive an appropriate version of the Fregean logicist project that remains faithful to the *Grundgestze* proposal.

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[3] M. Contente and A. Klev. *The Nature and Extent of Constructive Predicativity.* Presentation within CFvW Colloquium, The Carl Friedrich von Weizsäcker Center of the University of Tübingen, November 12, 2025.

[2] The classical approach of taking natural numbers for granted would not be a good starting point for the logicist programme of deriving arithmetic from a logical system augmented with some definitions.

[3] Ancestral of R: R*(xy) $\equiv_{def} \forall X((\forall z(Rxz \rightarrow Xz)) \wedge (Her(X,R) \rightarrow Xy))$ Weak Ancestral of R: $R^+(x,y) \equiv_{def} R^*(x,y) \vee x = y$. Predecessor: P(x,y) $\equiv_{def} \exists X \exists z(Xz \wedge y = \sharp X \wedge x = \sharp[Xw \wedge w \neq z])$. Natural Number: $\mathbb{N}x \equiv_{def} P^+(0,x)$.

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Fri, 14:35

Cut for Negative Free Classical Core Logic with Equality and a Variable-binding, Term-forming Operator

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Core Logic (henceforth $\mathbb{C}$) is an intuitionistic, relevant logic. As a natural deduction system, it differs from the traditional presentation of intuitionistic logic found in Gentzen [1] by requiring all elimination rules to be in parallel form, by requiring all major premises for elimination (henceforth "MPEs") to have no non-trivial proof work above them, and by omitting the rule EFQ. $\mathbb{C}$ is relevant because it proves neither the First nor the Second Lewis Paradox. That is, neither $A, \neg A \vdash_{\mathbb{C}} B$ nor $A, \neg A \vdash_{\mathbb{C}} \neg B$. Classical Core Logic (henceforth $\mathbb{C}^+$) extends $\mathbb{C}$ by adding one of the strictly classical rules of inference. Cut admissibility results for $\mathbb{C}$ and $\mathbb{C}^+$ were proved in [3] and [4], respectively. That is, it was proved that there is a procedure, which we call "the cut algorithm", that turns two proofs Π with conclusion A (called "the cut sentence") and Σ with conclusion θ of $\mathbb{C}$ (respectively $\mathbb{C}^+$) into a proof of either θ or $\bot$ from a subset of the premises of Π and Σ.

Here, we prove a similar result for the system Negative Free Classical Core Logic with Equality and a Variable-binding, Term-forming Operator (henceforth $\mathbb{C}^+_{=,@}$), the alteration of $\mathbb{C}^+$ which adds rules governing equality, adds rules governing the behavior of the symbol $\exists!t$ (an abbreviation of the sentence $\exists x\ x = t$), augments the quantifier rules by adding $\exists!t$ in logically salient places, and adds rules for a variable-binding, term-forming operator (henceforth VBTFO) of the form discussed in [2]. Examples of VBTFOs of this form include the definite description operator ι and the set-forming operator $\{ x \mid \dots x \dots \}$. Unlike other elimination rules in $\mathbb{C}^+_{=,@}$, the rule = elimination may have non-trivial proof work above its MPE.

To begin, we observe that there is a common, albeit highly abstract, form exhibited by all introduction rules and a corresponding form exhibited by all elimination rules in $\mathbb{C}^+_{=,@}$. We then inductively define the binary function $[\cdot, \cdot]$, which takes two proofs Π and Σ of $\mathbb{C}^+_{=,@}$ as arguments and returns a new proof. For the sake of brevity, we use the common forms for introduction and elimination rules in $\mathbb{C}^+_{=,@}$ to write the clauses of the definition of $[\cdot, \cdot]$. We then draw a distinction between two kinds of cut sentences: inessential cuts, in which the conclusion of Π is of the form $t = u$ and is either the conclusion of = introduction or @ introduction and is also the MPE for = elimination in the last step of Σ, and essential cuts, all other possible combinations of proofs Π and Σ. We then add a rule in which a proof ending in $t = u$ may be put atop the MPE of = elimination with the same form if and only if the conversion is an inessential cut.

We then define a cut algorithm for $[\cdot, \cdot]$ over the system $\mathbb{C}^+_{=,@}$ and show that the algorithm always terminates and produces a proof with only inessential cuts with either the same conclusion as the right argument of $[\cdot, \cdot]$ or $\bot$. We then prove several corollaries, including that a cut admissibility result holds for the intuitionistic subsystem $\mathbb{C}_{=,@}$ of $\mathbb{C}^+_{=,@}$ and that the proof work which stands above the MPE of = elimination can be restricted to only inessential cases without loss of deductive power. It is then proved, by reflecting on the general structure of

the introduction and elimination rules in $\mathbb{C}^{+}_{=,@}$, that cut admissibility results can be proved for any extension of $\mathbb{C}^{+}_{=,@}$ by rules governing any VBTFO of the form considered here.

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Sun, 13:45

Rigid Theories

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Motivation. A central theme in model theory is the study of structural invariance: a structure M is *rigid* if it admits no non-trivial automorphisms. When one passes from structures to theories, the natural morphisms are no longer automorphisms but *relative interpretations*, the structure-preserving transformations of the category of theories as studied by [6]. The self-interpretations of a theory T form a monoid under composition (up to homotopy); its structure measures the degree to which T can "re-read" its own language. We call T *rigid* if every self-interpretation is definably isomorphic to the identity.

There is a natural philosophical motivation. An influential tradition going back to Hilbert, Carnap, and Ajdukiewicz holds that the meaning of a formal theory is determined by its axioms. On this view, a self-interpretation represents an axiom-sanctioned alternative reading of the theory's own theorems. A rigid theory is therefore one whose axioms leave no room for alternative readings; rigidity provides a precise, purely syntactic criterion for what might be called the *semantic determinacy* of a theory.

Although the rigidity of theories appears to be a natural invariance criterion for theories, to the best of our knowledge, the triviality of self-interpretations in this general form was first investigated by [5], who showed, so to speak, that Presburger arithmetic (PrA) is rigid. In [3] we extend this investigation to other natural examples of first-order mathematical theories and aim for a more precise classification and analysis of theories with respect to this kind of determinacy, both philosophically and logically.

Definitions. We work with parameter-free, identity-preserving relative interpretations, allowing arbitrary dimensions. A formula $\eta(v_1, v_2)$ *defines a bijection* between the domain δ_f of an interpretation f and the universe of T if T proves η is a total bijective functional relation. We say f is *definably isomorphic to the identity* in T if such η exists satisfying, for every n-ary predicate symbol P:

$$T \vdash \forall v_1 \dots v_{2n}\Big(\bigwedge_{i=1}^{n} \eta(v_i, v_{n+i}) \to (Pv_1 \dots v_n \leftrightarrow f(P)v_{n+1} \dots v_{2n})\Big).$$

A theory T is *rigid* if every relative self-interpretation is definably isomorphic to the identity. A self-interpretation $f\colon T \to T$ is faithful, if $T \vdash \phi$ implies $T \vdash f(\phi)$ for all sentences ϕ; it is a *retraction* if there exists $g\colon T \to T$ s.t. $g \circ f$ is definably isomorphic to id_T in T.

Results. We establish rigidity for a range of mathematically natural theories.
Arithmetic.

Theorem 1. *Successor Arithmetic* SucA *and Skolem Arithmetic* SkA *are rigid in dimension* 1. *In dimension* ≥ 2*, interpreted models remain isomorphic to the standard model, but the isomorphism ceases to be definable.*

The proof for SucA uses the classification of countable models and the limited definability of functions in $(\mathbb{N}, 0, S)$. For SkA, one shows that the primes of any interpreted structure are prime powers in $\mathbb{N}$, then lifts the local Presburger isomorphisms (provided by the rigidity of PrA) to a global definable isomorphism.

Theorem 2. *Q has infinitely many faithful self-interpretations.* PA *has infinitely many faithful self-interpretations, but is rigid with respect to retractions.* $\mathrm{Th}(\mathbb{N})$ *is rigid.*

Non-rigidity of PA is witnessed using interpretability of inconsistency; infinitely many pairwise inequivalent self-interpretations arise from these; rigidity with respect to retractions follows from [7]. The rigidity of $\mathrm{Th}(\mathbb{N})$ uses a theorem of Feferman: an interpreted nonstandard model would have an arithmetically presented standard system contained in some Σ^0_k, contradicting the unbounded arithmetical complexity of subsets of $\mathbb{N}$.
Algebra.

Theorem 3. *The theory F_2 of non-trivial $\mathbb{F}_2$-vector spaces is rigid (dimension 1). The theories* RCF *and* ACF_0 *are rigid.*

For RCF, the key input is the Otero–Peterzil–Pillay theorem: every infinite field definable in an o-minimal expansion of a real closed field is definably isomorphic to the field or its algebraic closure. Since RCF rules out the latter, the isomorphism is to $\mathbb{R}$ itself, and its uniqueness (and parameter-freeness) follow from the rigidity of $\mathbb{R}$. For ACF_0, a theorem of Poizat reduces to definable subfields, which are shown to equal the whole model; uniqueness uses elimination of imaginaries.
Set theory.

Theorem 4. ZF *has infinitely many self-interpretations, but is rigid with respect to retractions. Every consistent extension T of* ZFC *is not rigid.*

Non-rigidity of ZF is similiar to PA. Rigidity with respect to retractions follows from [2]. The non-rigidity is witnessed by an internal iterated ultrapower; as in the construction of [4]. An internal Łoś theorem (proved within T) shows the resulting interpretation is a genuine self-interpretation of T. Then, there is no M-definable isomorphism between M and its interpreted model M_f, because from the point of view of M, M_f is not ω-standard.

Open question: Is there a rigid extension of ZF?

Related concepts. The rigidity of theories is logically independent of several related concepts that are often invoked as criteria for determinacy of theories: SucA and ACF_0 are $\aleph_1$-categorical and rigid; SkA and RCF are rigid without being categorical in any cardinality; DLO is $\aleph_0$-categorical, but not even rigid with respect to retractions. Model rigidity and theory rigidity are also independent; F_2 shows that rigidity is not restricted to complete theories. Furthermore, connections to currently discussed concepts of determinacy, such as solidity as in [1], can be discussed. We show that solidity implies rigidity with respect to retractions.

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From Herbrand schemes to functional interpretation

Sun, 14:10

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The conceptual connection between Herbrand's theorem and functional interpretation is well known. It was made explicit by Gerhardy and Kohlenbach in [5], in which Herbrand's theorem is proved using a functional interpretation of the classical first-order predicate calculus. The approach to Herbrand extraction via functional interpretation was further explored by Ferreira and Ferreira [4], who developed a so-called "Herbrandized" functional interpretation for classical first-order logic. In both these works, we obtain the result that the realizer extracted from a proof of an existential formula, when reduced to normal form, describes the witnessing set of terms for a Herbrand disjunction. Thus, Herbrand's theorem is elegantly located as a special case of functional interpretation.

These works form part of a quite large literature on various approaches to Herbrand's theorem. The present work builds on a framework called *Herbrand schemes*, developed by Afshari, Hetzl and Leigh [3], and further in [2, 1]. Herbrand schemes extract Herbrand disjunctions directly from sequent calculus proofs without relying on cut elimination. Functional interpretations, by contrast, are usually carried out for Hilbert-style axiomatic systems.

The idea of Herbrand schemes is to view a sequent calculus proof as a type of grammar, known as a *higher-order recursion scheme*. These grammars associate certain non-terminal symbols with formula occurrences in the end sequents of proofs, and these are supplied with rewrite rules depending on the last inference rule of the proof. For a proof of an existential formula, the language generated by the grammar is a set of witnessing terms for a Herbrand disjunction. Again, there is a connection with functional interpretation here, noted already in [3], in the use of higher-order types.

Here we aim to further complete the emerging picture by showing how, using ideas from Herbrand schemes, we can derive realizing terms in a typed lambda-calculus similar to that used by Gerhardy and Kohlenbach directly from proofs in classical sequent calculus. These realizers allow us to construct Herbrand disjunctions for valid existential formulas. We thus obtain again a method to extract Herbrand disjunctions directly from sequent calculus proofs without relying on cut elimination, but this time using functional interpretation instead of a higher-order grammar. The way that this term extraction works is very closely related to Herbrand schemes. In particular, we retain the following two features:

1. Each formula A is associated with two distinct types: an *evidence type* $[A]$ and a *counter-evidence type* $\langle A \rangle$ (called "output" and "input" types in [3]).

2. For each proof, and each formula occurrence in the end sequent, we construct a program that extracts evidence for that formula from potential counter-evidence for each formula occurence of the end sequent.

This program extraction is modular and pieces together programs extracted from immediate subproofs uniformly from the last inference rule used. In Herbrand

schemes, these programs are represented by non-terminal symbols of the grammar with associated rewrite rules. Here, we will similarly associate with each proof of end sequent $A_1, \ldots, A_n$ and index i a term of type $[A_i]$ parametric in given terms of types $\langle A_1 \rangle, \ldots, \langle A_n \rangle$; we will call this assignment a *term constructor*. In contrast with Herbrand schemes, however, there is no need to specify particular ad hoc rewrite rules for these terms; once the term has been constructed, its behaviour follows from the basic rewrite rules of the type theory. We note that the approach of assigning evidence and counter-evidence types to formulas has appeared before in connection with functional interpretation, in Pédrot's "functional functional interpretation" [6]. As far as we are aware, this approach has not previously been developed for classical logic and classical sequent calculus.

We believe this work provides further evidence of how natural the connection between functional interpretation and Herbrand's theorem is: it turns out that Herbrand schemes can be reformulated as a functional interpretation of sequent calculus proofs.

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A Proof-theoretic Foundation for Mathematics

Fri, 10:50

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The context and consequences of Gödel's incompleteness theorems for the foundations of mathematics in the early 20th Century are doubtless familiar. The orthodox view — and Gödel's own reading — is that there is a real world of arithmetic that outstrips our ability formalize it by finitary systems. Tarski [10] developed model-theoretic semantics as formal account of this view and it has been the dominant approach to semantics within logic for the last century. This is, however, not the only possible reading.

According to Dummett [1], this construal begs the very question incompleteness is supposed to bear on. To say that the consistency statement is *true but unprovable* presupposes a notion of arithmetical truth that floats free of provability, but it is precisely such truth that is at issue. The orthodox reading, therefore, does not *discover* a gap between truth and provability in some neutral way; rather, it builds one in from the outset. Dummett's own proposal was that the proper moral lies elsewhere. He argued that the concept of a ground for asserting something of all natural numbers is *indefinitely extensible*: any recursive characterisation of such grounds can be extended, by the Gödelian construction, to a strictly more inclusive one. The proposal attracted important objections — notably from Wright [11], Moore [7], and Raatikainen [8] — and in his Reply, Dummett [4] clarified that the principle of extension must itself admit of a precise characterisation, even though the totality it generates cannot.

In what follows I should like to suggest that recent developments in *proof-theoretic semantics* (P-tS) permit such a characterisation to be given, and that they thereby furnish a technically grounded reading of Dummett's proposal. The vehicle is the *base-extension semantics* (B-eS) for classical logic developed by Sandqvist [9]; see also Gheorghiu [5]. This semantics is itself presented as a candidate realisation of Dummett's wider programme for logic and mathematics.

Fix a first-order signature σ. An *atomic rule* is an inference figure

$$\frac{p_1 \quad \cdots \quad p_n}{C}$$

in which c and the p_i are closed atomic formulas ($\bot$ is not counted as atomic). A *base* $\mathfrak{B}$ is a set of atomic rules, and $\vdash_{\mathfrak{B}}$ denotes the relation of derivability obtained by closing the rules of $\mathfrak{B}$ under composition. The *support* relation $\Vdash$ is then defined inductively by the clauses in Figure 0.1; validity is given by $\Gamma \Vdash \varphi$ iff $\Gamma \Vdash_{\varnothing} \varphi$.

Classical derivability is sound with respect to support. It is also complete in the presence of an infinite reserve of constants disjoint from the theory under consideration; crucially, it may be incomplete without this reserve. The role played by such constants is, in effect, that of eigenvariables; and the requirement that they be available is, philosophically speaking, the requirement that the language be open to extension. I argue that this gap is where we ought to understand the Gödelian incompleteness phenomenon to be about.

Dummett [3] believed that logic was determined by a theory of meaning, which was itself determined by a set of practices for justifying the assertion of expressions. This view applied to logic arguably delivers the B-eS above. When applied

$\Vdash_{\mathfrak{B}} p$	iff	$\vdash_{\mathfrak{B}} p$	(At)
$\Vdash_{\mathfrak{B}} \varphi \to \psi$	iff	$\varphi \Vdash_{\mathfrak{B}} \psi$	($\to$)
$\Vdash_{\mathfrak{B}} \forall x \varphi$	iff	$\Vdash_{\mathfrak{B}} \varphi[x \mapsto t]$ for all closed terms t	($\forall$)
$\Vdash_{\mathfrak{B}} \bot$	iff	$\Vdash_{\mathfrak{B}} p$ for every closed atom T	($\bot$)
$\Delta \Vdash_{\mathfrak{B}} \varphi$	iff	for all $\mathfrak{C} \supseteq \mathfrak{B}$, if $\Vdash_{\mathfrak{C}} \psi$ for all $\psi \in \Delta$, then $\Vdash_{\mathfrak{C}} \varphi$	(Inf)

Figure 0.1: Support in a Base

to a first-order theory T, we may read it as saying that the ontology is fixed not by reference to an antecedently given domain but by the inferential practice licensed by T for the closed terms within it. That is, an *inferential ontology* is determined by the signature σ — as it specifies the closed terms — together with T — as it specifies their inferential roles.

Two cases must therefore be sharply distinguished. In the signature $\sigma := \langle 0, S, +, \cdot \rangle$ in which arithmetic is ordinarily formulated, if the theory T is such that every closed term is equivalent to a numeral; hence, the ontology is exhausted by the numerals. In a richer signature with additional constants $(c_i)_{i \in \mathbb{N}}$, by contrast, this is may not be the case. A constant c for which, using only the inferences of T, we cannot assert to be equal to a numeral plays the role of a non-standard number. These two setups have very different properties.

Working in the standard signature σ, let us say that a theory T is *sufficiently strong* if it decides Δ_0-sentences and identities between numerals — conditions met by Robinson's Q and by PA alike — and let $\mathrm{Prov}_T(x)$ be the standard provability predicate for T. The reflection schema then obtains:

$$T \Vdash \mathrm{Prov}_T(\ulcorner \varphi \urcorner) \to \varphi \qquad \text{for every sentence } \varphi.$$

Taking φ to be $\bot$ gives $T \Vdash \mathrm{Con}(T)$. The proof follows the structure of a naïve internal soundness proof using the fact that the ontology is exhausted by the numerals: any putative witness to $\mathrm{Prov}_T(\ulcorner \varphi \urcorner)$ must be a numeral coding an actual T-derivation, and one then proceeds by induction on the length of that derivation — full details are given in [6]. In the enriched setting with additional constants, the naive soundness proof does not go through as the ontology outruns the coding.

Two morals follow. First, in the finite signature in which arithmetic is naturally formulated, derivability and support come apart, but they do so *within a single theory* rather than as a divergence between proof and an externally given structure. On the present framework, then, Gödel's second incompleteness theorem is explicable as a consequence of the ontology's being fixed by the theory itself. In a richer signature with extra constants, where nonstandard witnesses become available, classical completeness is recovered and reflection fails accordingly — as it must, on pain of contradicting Gödel. Second, the reflection schema $\mathrm{Prov}_T(\ulcorner \varphi \urcorner) \to \varphi$ is precisely such a general characterisation of the principle of extension as Dummett insisted upon in the reply to Wright. To assert the premiss $\mathrm{Prov}_T(\ulcorner \varphi \urcorner)$ is to perform the act of extension, not to describe a number already present in some antecedently fixed domain; and the indefinite iterability of the schema is what ensures that no recursive axiomatisation can capture its closure.

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Undecidability in Relevant Logic

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Initially presented proof-theoretically, the problem of providing a model theory for relevant logics led Urquhart [16, 17] to develop semilattice semantics. In the signature $\{\to, \wedge\}$, the resulting semilattice logic $\mathsf{S}_{\to,\wedge}$ enjoys the finite model property, is decidable, and coincides with $\mathsf{R}_{\to,\wedge}$, the implication-conjunction fragment of the relevant logic R. In the presence of disjunction, the logic $\mathsf{S} = \mathsf{S}_{\to,\wedge,\vee}$ properly extends its R-counterpart $\mathsf{R}^+ = \mathsf{R}_{\to,\wedge,\vee}$. While S is known to be recursively enumerable and was axiomatized by Fine [7] and Charlwood [4], it has been an enduring open problem whether it is decidable.

More broadly, decision problems for relevant logics have been studied since the beginnings of the field, with an early mention in Curry and Craig's [6] review of Church's [5] weak theory of implication, which corresponds to the implication fragment $\mathsf{R}_{\to}$ of R. Kripke [13] proved $\mathsf{R}_{\to}$ decidable, and Meyer [15] extended the result to the implication-conjunction fragment $\mathsf{R}_{\to,\wedge}$. Still, the decision problem for R^+ (let alone for R) remained open for another two decades, until Urquhart [18] settled the matter. Using a Lipshitz–Hutchinson-style [14, 10] reduction that passes via a coordinatization theorem of von Neumann [20], he showed that many relevant logics, including R^+, E^+ and T^+, are undecidable. Intriguingly, as noted in [18], S eluded these techniques, and its decision problem remained unresolved. Eventually, this led experts, including Urquhart [19], to conjecture that S is decidable. Contrary to this conjecture, we show that S is undecidable. In doing so, we prove undecidability for a broad class of relevant logics, among which are R^+, E^+ and T^+ (to the author's knowledge, the proof of Urquhart [18] remains the only prior undecidability proof for these systems).

The weakest system we prove undecidable is the relevant logic TWJ^+. It extends the logic TW^+ (which, in contrast, is decidable, cf. Giambrone [9]) by the axiom for hypothetical syllogism

$$[(\varphi \to \psi) \wedge (\psi \to \chi)] \to (\varphi \to \chi).$$

TW^+, in turn, extends the minimal basic relevant logic B^+ by the axioms for prefixing and suffixing

$$(\varphi \to \psi) \to [(\chi \to \varphi) \to (\chi \to \psi)] \quad \text{and} \quad (\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)].$$

At a more conceptual level, our aim is to better understand the boundary between the solvable and the unsolvable. To this end, our proof identifies these axioms, in combination, as a source of undecidability and explains the mechanism by which they lead to it.

The proof is by reduction from the Wang tiling problem, introduced by Wang [21, 22] in the study of the decidability of the $\forall\exists\forall$-fragment of first-order logic and proven undecidable by Berger [3]. A Wang tile is a square with coloured sides and a fixed orientation (no rotation or reflection allowed). Given a finite set of tiles $\mathcal{W}$, the problem asks whether the plane can be covered with translated copies of

tiles from $\mathcal{W}$ placed side-by-side so that adjacent tiles have matching colours on shared sides.

To code the tiling problem, for each finite tile set $\mathcal{W}$, we construct a formula $\psi_{\mathcal{W}}$ and prove that

(i) if $\mathsf{TWJ}^+ \nvdash \psi_{\mathcal{W}}$, then $\mathcal{W}$ tiles the plane, and

(ii) if $\mathcal{W}$ tiles the plane, then $\mathsf{S} \nvdash \psi_{\mathcal{W}}$.

We shall in fact prove a stronger statement than (ii), but (ii) as stated suffices to conclude that every logic L with $\mathsf{TWJ}^+ \subseteq \mathsf{L} \subseteq \mathsf{S}$ is undecidable (which, besides TWJ^+ and S, also includes T^+, E^+, and R^+), since for such L, (i) and (ii) imply

$$\psi_{\mathcal{W}} \notin \mathsf{L} \qquad \text{if and only if} \qquad \mathcal{W} \text{ tiles the plane.}$$

It is worth noting that we do not prove (i) directly. Instead, we show that if $\mathsf{TWJ}^+ \nvdash \psi_{\mathcal{W}}$, then $\mathcal{W}$ tiles isosceles right triangular regions of the plane of arbitrarily large finite size, which, by a Kőnig's Lemma argument, implies (i).

An earlier version of this work appeared as the conference paper [11]; this abstract describes an extended journal version in preparation that simplifies the proof, generalizes the result, and explains the tiling intuitions in greater depth. The latter is also relevant in view of the apparent breadth of the methods developed here and elsewhere, including in [12, 8] where related tiling-based techniques are applied in modal logic and bunched implication logic.

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From Tonk to Harmony via Identity and Cut

Fri, 11:15

Barteld Kooi[a] and Allard Tamminga[b]

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Prior's rebel connective *tonk* forced logicians to rethink the systemic (that is, syntactic or semantic) background conditions that are to be preserved under the addition of a novel connective to a logical language. By now, it has become clear that *tonk* is not the only rebel attacking the conservativeness of the system. Other rebel connectives, some with more and some with less destructive force than *tonk*, wreaking havoc in a variety of ways, have been deployed to expose the necessities of the logical establishment.

Rather than conjuring up a novel rebel connective, comparing the disruptive power of the various rebel connectives already in existence, or pointing out how crazy the world would become if a particular rebel connective got its way, in this paper we arrive at general results that hold for all connectives via a thorough study of *tonk* itself. We do so within the logical framework of classical propositional truth-functional validity. Central to our concerns are the concepts of gappiness and gluttiness. Using Quinean quotation marks for readability only, let $\mathrm{S} = \{\ulcorner f(x,y) = z \urcorner : x, y, z \in \{0,1\}\}$ be the set of all possible binary two-valued truth table entries. (Note that an entry is a statement that can be true or false.) Let $S^\star \subseteq \mathrm{S}$ be a set of entries for a binary connective $\star$. We say that $S^\star$ is *gappy* if there are $x, y \in \{0,1\}$ such that both $\ulcorner f(x,y) = 0 \urcorner$ and $\ulcorner f(x,y) = 1 \urcorner$ are *not* in $S^\star$. We say that $S^\star$ is *glutty* if there are $x, y \in \{0,1\}$ such that both $\ulcorner f(x,y) = 0 \urcorner$ and $\ulcorner f(x,y) = 1 \urcorner$ are in $S^\star$. In Section 2 we show that Prior's *tonk* is characterized by a set S^τ of entries that is both gappy and glutty.

In the sections that follow we study the proof-theoretical repercussions of gappiness and gluttiness in sequent systems. In Sections 3 and 4, we define for every set $S^\star$ of entries a Gentzen system $\mathbf{G}_{S^\star}$ that characterizes it, and in Section 5 we use this to study *tonk*. We then establish two main results. In Section 6 we show that $S^\star$ is not gappy if and only if $\mathbf{G}_{S^\star}$ admits *Identity*-elimination, and in Section 7 we show that $S^\star$ is not glutty if and only if $\mathbf{G}_{S^\star}$ admits *Cut*-elimination. Moreover, in Section 8 we show, first, that $S^\star$ is not gappy if and only if $\mathbf{G}_{S^\star}$ is complete (but not necessarily sound) with respect to the semantics induced by $S^\star$ and, second, that $S^\star$ is not glutty if and only if $\mathbf{G}_{S^\star}$ is sound (but not necessarily complete) with respect to the semantics induced by $S^\star$.

Returning to our initial observation that the set S^τ of entries that characterizes Prior's *tonk* is gappy and glutty, we conclude that $\mathbf{G}_{S^\tau}$ is neither sound nor complete and admits neither *Identity*-elimination nor *Cut*-elimination. Moreover, with the results from this paper we can conjure up three types of *tonk*-like connectives, depending on whether the sets $S^\star$ of entries that characterize them are gappy (but not glutty), glutty (but not gappy), or both gappy and glutty. On the one hand, it is clear that, first, every set of sequent rules that is characterized by a gappy set of entries suffers from one sort of mismatch and, second, every set of sequent rules that is characterized by a glutty set of entries suffers from another sort of mismatch. On the other hand, sets of sequent rules that are characterized by a

set of entries that is neither gappy nor glutty do not suffer from these two sorts of mismatches. We thus submit that the following definition of proof-theoretical *harmony* is natural.

Definition. Let $S^\star \subseteq \mathrm{S}$. Then the rules in $\{R^\star_{xyz} : \ulcorner f_\star(x, y) = z \urcorner \in S^\star\}$ are *harmonious* with respect to **G** if and only if $S^\star$ is neither gappy nor glutty.

Observation. Let $S^\star \subseteq \mathrm{S}$. Then the following are equivalent:

(i) The rules in $\{R^\star_{xyz} : \ulcorner f_\star(x, y) = z \urcorner \in S^\star\}$ are *harmonious* with respect to **G**;

(ii) $\mathbf{G}_{S^\star}$ admits *Identity*-elimination and *Cut*-elimination;

(iii) $\mathbf{G}_{S^\star}$ is sound and complete.

As long as our logic is a propositional logic and our concept of validity is classical truth-functional validity, this concept of harmony seems to do the job. It excludes Prior's *tonk* (and related connectives), and it includes every standard truth-functional connective.

Lastly, our findings on harmony show that, within the context of classical propositional logic, there is a precise relation between how non-defective truth conditions for a given operator must be formulated and how non-defective introduction and elimination rules for that operator must be formulated. At present, we do not know whether our findings on the relation between a truth-based account of meaning and a rule-based account of meaning are scalable, but at the very least they show that the supposed rift between theories of meaning based on truth-conditions and theories of meaning based on usage is not as deep and principled as many authors assume.

From a Computability-Theoretic Point of View: The Real Numbers and Their Representations

Sun, 14:35

Lars Kristiansen
University of Oslo

There are many ways to represent real numbers. All mathematicians will be familiar with classic representations like decimal expansions and Cauchy sequences and some, also with continued fractions and other representations related to the Stern-Brocot tree. There are also representations of a more recent date, e.g., "nested intervals" which is a popular representation in modern computable analysis.

Mathematicians have, through the years, invented and worked with quite a few different representations of the real numbers. The reason might be that a representation that works well in one context does not work well in another. I will consider all sorts of representations and discuss problems we might run into when we want to convert one representation into another. I will conclude the talk with a summary of some of my own research (see the references below).

This will be a non-technical talk aimed at a general audience of mathematicians, computer scientists, and philosophers.

[1] Amir Ben-Amram, Lars Kristiansen, and Jakob G. Simonsen. *On representations of irrational numbers and the computational complexity of converting between such representations.* ***Bulletin of Symbolic Logic***, 2025. doi: 10.1017/bsl.2025.10085.

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Sat, 10:25

Partition-Based Topic-Sensitive Intentional Modals

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In recent years, Berto [1] and his collaborators have developed a framework for topic-sensitive intentional modals (TSIMs), which uses algebras of topics to make intentional modal operators sensitive to the topics of embedded formulas. In this talk, I argue that this framework can be simplified and improved by modeling topics as partitions of logical space. After reconstructing the algebra-based semantics and its core principles, I raise three objections: first, it lacks a principled connection between truth conditions and topics; second, it renders TSIMs topic-transparent, incorrectly predicting that they do not affect topical content; third, it characterizes topics only by their formal behavior, rather than explaining what topics are. I then develop an account of topics that refines Lewis's ([2]) partition-based approach and use it to formulate a new semantics for TSIMs. The resulting framework preserves the core principles of the algebraic approach while establishing a principled connection between truth conditions and topics, avoiding topic-transparency, and explaining what topics are.

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Atomic steps of computation and β-reductions

Sun, 15:20

Chirine Yasmine Laghjichi
LIPN-Université Sorbonne Paris Nord

It has been a long time concern, that the notion of β-reduction as a step of evaluation in the model of λ-calculus cannot be considered as an atomic step of computation in the sense of a unit cost step of computation. This question was raised in the paper of Lafont [7] in 1997, and is a very central motivation for the work of Accattoli [3] in 2019 and rephrased in the very recent paper and work of Balabonski [5] in 2025. This question is essential for the theory related to the complexity of λ-calculus in order to define a cost model for λ-calculus. This issue is in the meantime central for the philosophical analysis of λ-calculus as a model of computation. Indeed, differently from its machine-base counterparts like the Turing machine, or the RAM machine, the dynamics of the computation of a λ-term depends on the choice of a particular reduction strategy. This leads to different accounts of how λ-calculus actually carries out the steps of a computation and this hinders a unified approach to its role as a model of computation in the same sense as the other instantiations mentioned previously. The mitigated relation towards λ-calculus as a true model of computation with an accurate and realistic cost model assigned to it seems to derive from its lack of a clearly defined notion of atomic steps of computation. Moreover, as we are going to show in this talk, the most common attempts at solving this problem draw on mechanistic resources, like abstract machines that are highly local procedures, but also draw away from the inherently machine-independent nature of λ-calculus.

The way in which we are going to analyse this broad issue is through the analysis of Accatoli [1], [2], [3], and collaborations with Dal Lago such as [4], which emphasize the importance of the size-explosion problem. This is a phenomena in which a lambda-term, who's evaluation length is linear to its size, however evaluates to an intermediate term that is exponential to its initial size. This means that, one intermediate evaluation step takes more time to write down than the whole evaluation itself, which means that the cost model that is based solely on the number of evaluation steps is not well suited for these terms.

We claim that a new and relevant perspective towards this issue is the one who focuses on the condition of locality. We take it as being an essential condition for machine based models of computation, as setting a restriction on the transition function of the system and determining an atomic step of computation. This restriction bounds the amount of information that is taken as argument for each transition function. However, according to Wilfried Sieg [8], this condition of locality which is essential to machine models makes "no sense" for functional approaches to computation. This might be understood from the fact that λ-calculus isn't defined from a transition function, as it is rather a rewriting system. Moreover, according to Sieg, the "computation" of λ-calculus is reduced to a number of rewriting rules with no accurately defined condition of "effectiveness". In other words in depends on a choice, or on a particular strategy that needs to be added in order to actually carry out the computation, or the evaluation process of a λ-term. For these reasons, λ-calculus is not making global consensus regarding

its definition as a model of computation although it is presented as such in modern textbooks of computer science. It inherited that qualification from its role within the Church-Turing thesis (CTT), as being equivalent to the other models of Gödel and Turing. This is the CTT's extensional equivalence, but when the attention is shifted towards the procedure of computation itself however, the matters are different. Our analysis aims at questioning the assumption of Sieg and claims that the attempts at defining an elementary step of computation for λ–calculus, i.e., its effectiveness, requires a condition of locality explicit at the level of implementation.

In the first part of our discussion we will show in which sense the "abstract machines" of Krivine [6], implementing λ-calculus by fixing a call-by-name reduction strategy, considers a condition of locality by explicitly defining the states and transitions of the system. This can be seen in the manipulation of a memory that restricts by its addressing the access of execution rules to the stack. However, this approach is still mechanistic, and does not allow a true analysis of λ-calculus without passing through its operational semantics. We will continue with the account of Accattoli, aiming at overcoming the mechanistic frontier, and which develops the notion of "sharing". This allows to derive the cost model from the number of micro-steps of the reduction strategy, the size-explosion problem not arising at this level of analysis. We will then end the presentation by exploring a relevant and new machine-independent proposition for defining a cost model for λ-calculus by Balabonski. He reformulates the problem of the atomic step of computation by emphasizing the impossibility of restricting the number or variables undergoing the reduction, which is the consequence of an essential property of λ-calculus, that the inputs can't be distinguished from the terms of the formula. In order to do so he introduces the 'history of computation' which is the search of the antecedent and that allows a treatment of the intermediary steps.

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[2] Beniamino Accattoli. *The Complexity of Abstract Machines.* ***Electronic Proceedings in Theoretical Computer Science***, 235:1–15, 2017.

[3] Beniamino Accattoli. *A Fresh Look at the λ-Calculus.* In ***FSCD 2019 – 4th International Conference on Formal Structures for Computation and Deduction***, Dortmund, Germany, 2019.

[4] Beniamino Accattoli and Ugo Dal Lago. *(Leftmost-Outermost) Beta Reduction is Invariant, Indeed.* ***Logical Methods in Computer Science***, 12(1):1627, 2016.

[5] Thibaut Balabonski. *A Machine-Independent, Log-Sensitive Space-Cost Measure for the Weak Lambda-Calculus.* hal-05322360, 2025.

[6] Jean-Louis Krivine. *A call-by-name lambda-calculus machine.* ***Higher-Order and Symbolic Computation***, 20(3):199–207, 2007.

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Semantic Plurivalence is Logically Idempotent

Sat, 10:50

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This talk intends to present the content of a forthcoming paper with the same title [4] and the following abstract can be found as most of the introduction of this paper.

A common reason to introduce many-valued semantics for logics is to allow formulas to admit truth-states that can be understood as ambiguously taking multiple basic truth values simultaneously. An important exaple is the logic of paradox (LP) by Priest [6], for which a third truth value is admitted to allow formulas to be interpreted as *both* true and false simultaneously without making the logic inconsistent. An immediate concern with this way of resolving paradoxes is that LP may admit its own contradictions for which further overlapping truth-values (e.g being both *true* and *both* simultaneously) needs to be introduced. This regression could inevitably lead to the need of infinitely many truth values of all possible simultaneous truth-states considered simultaneously as truth-states on their own ([5] cited in [7]).[1] However, in [7] Priest showed that any further repeating of his construction for the semantics of LP, admitting higher order of simultaneous truth-values, always results in a system that is logically equivalent to LP, so that the possibly infinite regression logically collapses to the first stage, and thus the three valued semantics for LP is adequate to handle also these higher order notions of paradox.

After proving similar results for other particular logics, Priest generalised his construction of admitting multiple truth values in what he calls *Plurivalent semantics* [8]. In doing so he exemplifies the construction by presenting it for the *FDE-family* of logics, being a collection of eight logics definable from the same algebraic matrix, for which most are household names in the many-valued logic scene. For this specific family of logics he also presented the same collapsing result for any repetition of the construction, by showing that plurivalence can be mimicked by the additional inclusion of a specific element in the underlying algebra. If this element already is included however, the construction is conservative.

In [8], Priest expressed his belief that his general construction of plurivalence is new. However, as also directly pointed out by Humberstone [3], similar lifting of algebraic operations from elements to subsets has rendered some previous attention, most noticeably in a logical context by Brink [1] under the name of *power-algebras*. Even though Brink and Priest define the same algebraic structures, they differ in how they interpret them for logical applications. Brink considers logics defined in terms of pre-orders on algebras, and thus focuses his attention on identifying and comparing appropriate partial orders definable on the power-lifted algebras. Priest, on the other hand, considered logics in terms of logical matrices, being an algebra with a designated subset of satisfactory elements. Priest's construction, when interpreted in terms of power-algebras, thus pairs the power-lifting of the algebra with a lifting of the designated set, forming what Humberstone names a *power-matrix* of the original matrix [3].

[1] This was a reasonable concern, as it would mimic similar worries considered in the early days of the development of intuitionistic logic [2].

In this talk, after recounting the notion of algebraic matrix semantics and power-algebras, we fully describe Priest's plurivalent constructions in terms of a notion of the power-matrix of an algebraic matrix. We can then support the correctness of these constructions by observing that standard set-theoretic mappings are homomorphic mappings between matrices and their power-lifts. We also confirm that homomorphism between algebraic matrices are naturally lifted to homomorphisms between their power-matrices. As a direct consequence, we establish a full generalisation of Priests collapsing result to arbitrary many-valued logics, proving that any number of power-liftings of matrices gives a matrix that is logically equivalent to the matrix of a single lift.

The main contribution presented is the fully algebraic formulation of the plurivalent constructions, allowing for an algebraic formulation of the then almost self evident desirable theorems. This work also initiates further investigations of plurivalence through abstract algebraic means, in particular for plurivalence and power-lifting in terms of classes or categories of algebraic matrices.

[1] Chris Brink. *Power structures.* ***Algebra Universalis***, 30(2):177–216, 1993.

[2] Alonzo Church. *On the law of excluded middle.* ***Bulletin of the American Mathematical Society***, 34(1):75–78, 1928.

[3] Lloyd Humberstone. *Power matrices and Dunn–Belnap semantics: Reflections on a remark of Graham Priest.* ***The Australasian Journal of Logic***, 11(1):14–45, 2014.

[4] Orvar Lorimer-Olsson. *Semantic Plurivalence is Logically Idempotent.* To appear in the proceedings of the ESSLLI 2026 Student Session, 2026.

[5] Robert K. Meyer. *Why I am not a relevantist.* Research paper #1, Logic Research School of Social Sciences, Australian National University, 1978.

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Should we worry about Duplication?

Sat, 11:15

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Algebra-valued semantics for set theory generalizes the familiar Boolean-valued framework by allowing truth values to range over broader algebraic structures. In the standard approach, however, one does not merely vary the underlying algebra; one also fixes interpretations of membership and equality. Under the usual interpretation, equality is defined extensionally, in terms of agreement in membership behavior. This provides a natural algebraic reading of extensionality, but it also gives rise to a characteristic phenomenon already present in Boolean-valued models: duplication. Distinct names may count as equal inside the model.

In classical settings, this phenomenon often appears harmless. Beyond the Boolean and Heyting cases, however, duplication can become problematic: key properties required for a mathematically expressive model may fail, including Leibniz's Law and the expected behavior of bounded quantification. The failure of Leibniz's Law, for example, prevents us from quotienting the model by internal equality and thereby obstructs the development of a well-behaved mathematical ontology. The failure of the bounded quantification principles, on the other hand, significantly complicates the evaluation of set-theoretic statements involving bounded quantifiers.

This paper asks whether algebra-valued semantics can be made genuinely more extensional by replacing the standard interpretation of equality with a stricter one: true identity, where equality in the model is just literal meta-theoretic identity of names. The guiding thought is simple. If duplication is the problem, perhaps the right remedy is to stop identifying distinct names altogether. I examine this proposal by comparing the standard and true-identity interpretations on the same algebra-valued universes, across a broad range of algebraic structures. The central claim is that this anti-duplication strategy produces a striking reversal: the interpretation that looks cleaner and more extensional turns out to be mathematically weaker, while the interpretation that tolerates duplication turns out to support richer set-theoretic structure.

We shall argue that the broader philosophical lesson is that duplication is not a dispensable defect of algebra-valued models. In non-classical settings, it is better understood as the visible sign of an irreducibly intensional aspect of names. Eliminating duplication therefore does not yield a genuinely more extensional semantics. It merely relocates the same underlying phenomenon.

Announcing Alternatives

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In this work, I extend inquisitive logic with resources from Public Announcement Logic (PAL) to capture the effects of announcing answers to questions. Inquisitive logic [1, 2] features formulas that express questions, in addition to formulas that express statements. Both questions and statements are given a uniform semantics in terms of support at information states. The least informative information states supporting a formula are called its *alternatives*. Any formula that expresses a question is associated with multiple alternatives, which can naturally be understood as *minimally informative answers*: answers that provide just enough information to settle the question. In this work, I introduce an extension of inquisitive logic called *Alternative Announcement Logic* (AAL). AAL features dynamic operators that capture the effects of updating models with alternatives. Formulas of the form $[\varphi]\psi$ express that ψ is supported after updating with an alternative for φ. If φ and ψ express statements, the meaning of $[\varphi]\psi$ essentially boils down to the standard PAL interpretation. On the other hand, if φ expresses a question, $[\varphi]\psi$ intuitively means that ψ is supported after updating with a minimally informative answer to φ.

Propositional inquisitive logic is a conservative extension of classical propositional logic that can express both statements and questions. Its language includes a connective $\vee$ called *inquisitive disjunction*, which can be used to formalize questions. For example, $p \vee \neg p$ expresses the question *whether* p. The language of AAL extends the language of propositional inquisitive logic with new types of modal formulas. Apart from dynamic formulas of the form $[\varphi]\psi$, the language also contains a modal operator $\Box$, which is intended to express the knowledge of an implicit agent, and an inquisitive version of the global modality included for technical reasons.

Models are structures $\mathcal{M} = (W, R, V)$ where W is a set of worlds, $R \subseteq W \times W$ and V is a valuation function. Any subset $s \subseteq W$ is called an *information state*. As is standard in inquisitive logic, semantics is defined in terms of a relation of *support* $(\mathcal{M}, s \models \varphi)$ relative to a model $\mathcal{M}$ and an information state s. In particular, inquisitive disjunctions $\varphi \vee \psi$ are supported if one of the disjuncts is supported. The set $\mathrm{alt}_{\mathcal{M}}(\varphi)$ of *alternatives* for a formula φ in a model $\mathcal{M}$ is defined as the maximal states (in terms of the subset relation) that support φ in $\mathcal{M}$. While formulas that express statements are associated with a single alternative, formulas that express questions are associated with multiple alternatives; as mentioned previously, these are naturally thought of as the questions' minimally informative answers. The semantics for dynamic formulas is defined as follows:

- $\mathcal{M}, s \models [\varphi]\psi$ iff for all $t \in \mathrm{alt}_{\mathcal{M}}(\varphi)$, if $t \neq \emptyset$ then $\mathcal{M}|_t, s \cap t \models \psi$

where $\mathcal{M}|_t$ is the restriction of $\mathcal{M}$ to the information state t. If φ and ψ express statements, the semantics of $[\varphi]\psi$ essentially boils down to the standard interpretation in PAL. On the other hand, if φ expresses a question, $[\varphi]\psi$ expresses that ψ is supported after updating with a minimally informative answer to φ. For example,

consider the formula $[p \mathbin{\mathbb{V}} \neg p]q$, which expresses that q is supported after updating with a minimally informative answer to the question *whether* p. Provided that $\mathcal{M}$ contains worlds where p is true and worlds where p is false, the alternatives for $p \mathbin{\mathbb{V}} \neg p$ in $\mathcal{M}$ are the truth sets of p and $\neg p$. In such a model, $[p \mathbin{\mathbb{V}} \neg p]q$ turns out to mean the same thing as $[p]q \wedge [\neg p]q$.

An interesting feature of AAL is that it can be used to reason about *auxiliary questions* that may be relevant when inquiring into a main question. For example, if φ and ψ express questions, then the formula $[\varphi]\Box\psi$ expresses that by updating with a minimally informative answer to φ, ψ is known afterwards. Thus, $[\varphi]\Box\psi$ can be used to express that φ is a relevant auxiliary question given that one is inquiring into ψ: if $[\varphi]\Box\psi$ holds, one's inquiry into ψ is resolved once the auxiliary question φ is answered.

AAL is axiomatized using reduction axioms. While some of the standard reduction axioms for PAL generalize straightforwardly to the inquisitive logic setting, the natural generalizations of reduction axioms for implication and modal operators fail: $[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$ and $[\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi)$ are not sound when φ contains occurrences of inquisitive disjunction. I adapt techniques from recent work on inquisitive conditional logic [3] in order to handle the interaction between inquisitive disjunction and dynamic operators, making essential use of the inquisitive version of the global modality.

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Logical Modeling of Belief Polarization

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Polarization as a social phenomenon has been described and studied in many forms. Several sources point to different subcategorizations of the phenomenon, including, among others, *belief polarization*. According to [4], belief polarization occurs when people who disagree on a proposition individually strengthen their own beliefs when presented with a mixed body of evidence which both confirms and contradicts the belief they already hold. This should not be confused with group polarization, which also involves deliberation within the group, or with political polarization, which refers to division into ideological poles in democratic societies [4]. Recent works, such as [11, 8, 7], have developed modal logics to reason about group polarization, but little attention has yet been paid to exploring belief polarization at the individual level by utilizing the tools of epistemic and doxastic modal logics. In this project, we begin to bridge this gap by discussing ways of modeling information updates in which people update their beliefs with respect to the bias that they already have.

Our proposal uses several ingredients coming from dynamic epistemic logic [2], such as updating evidence in a setting based on evidence models [9], plausibility models [3], and weighting models [1, 6]. In our setting, agents' beliefs are tied to the evidence they have at hand, thus changing the strength of evidence changes the agents' beliefs. We are interested in both quantitative and qualitative updates; in the following, we sketch a proposal for a quantitative update and discuss what a similar qualitative update could entail.

We begin by recalling the weighting models first introduced in [6] and further developed in [1]. Define At to be a countable set of atomic propositions. A **weighting model** is a structure $\mathcal{M} = (S, E, f, ||\bullet||)$ where S is a finite set of states; $E \subseteq \mathcal{P}(S)$ is a family of non-empty subsets $e \subseteq S$ ($\emptyset \notin E$) called evidence (sets) such that S is itself an evidence set ($S \in E$); $f : E \to \mathbb{N}$ is a function that assigns natural numbers to evidence sets; and $||\bullet|| : \text{At} \to \mathcal{P}(S)$ is a valuation map. The intuition behind the weighting model is that it models what evidence (modeled as sets of possible worlds) an agent considers possible. The function f assigns a number to each evidence set denoting how convincing the evidence is to the agent. Consider the example in Figure 0.2. In this model, we have $S = \{w, v, s, t\}$ and evidence sets $E = \{e_1, e_2, e_3, e_4, S\}$ where the agent considers e_1 and e_2 the most convincing evidence and S the least convincing evidence according to the function f.

As defined in [1], the function f can be extended to the family of all bodies of evidence $\varepsilon \subseteq \mathcal{P}(E)$ such that $f(E) = \sum_{e \in E} f(e)$. Furthermore, define the **largest body of evidence** consistent with a given state $s \in S$, written $E_s := \{e \in E \mid s \in e\}$. Now, define the ordering for all $s, t \in S$: $s \leq_E t$ iff $f(E_s) \leq f(E_t)$. In [1], $\leq_E$ is shown to be a total plausibility ordering on S, meaning that the weighting model induces a total plausibility model as defined in [3]. For all $s, t \in S$, let $s <_E t := s \leq_E t \,\&\, t \not\leq_E s$ and $s \simeq_E t := s \leq_E t \,\&\, t \leq_E s$. Returning to the example, note that $w \simeq_E v <_E t <_E s$, intuitively read as the agent considers states w and v equally

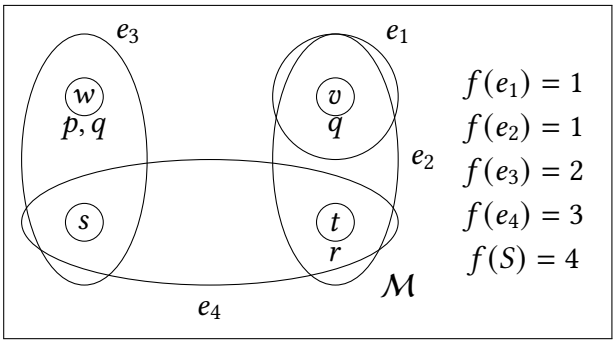

Figure 0.2: A weighting model $\mathcal{M}$

most plausible, followed by t and s.

Following [3, 10], we define the **maximal elements** of a set $X \subseteq S$ denoted $Max_{\leq_E} X := \{s \in X \mid s \leq_E s' \text{ for all } s' \in X\}$. We also denote $P = ||p||$ as the truth set of $p \in \text{At}$. Then, following [1] we can define the epistemic and doxastic notions: $KP := \{s \in S \mid P = S\}$; $BP := \{s \in S \mid Max_{\leq_E} S \subseteq P\}$; $B^Q P := \{s \in S \mid Max_{\leq_E} Q \subseteq P\}$; $S_b P := \{s \in S \mid P \neq \emptyset \text{ and } (\forall t \in P \text{ and } \forall w \notin P) : w <_E t\}$; and $K_D P := \{s \in S \mid t \not>_E s \Rightarrow t \in P\}$. We read KP as "the agent knows P", BP as "the agent believes P", $B^Q P$ as "the agent believes P conditional on Q", $S_b P$ as "the agent strongly believes P" and $K_D P$ as "the agent has defeasible knowledge that P".

Next, we propose a quantitative evidence update operation that is novel in this work. We are motivated to define a model update in which the agent updates their beliefs with respect to the bias that they already have. Thinking of weighted evidence as capturing the agent's biases, we can define an update on weighthing models that updates only the function f. Consider the updates $p\uparrow$ and $p\downarrow$ for a given atomic proposition p. The idea is that after an update $p\uparrow$, p becomes more convincing, whereas after an update $p\downarrow$, p becomes less convincing. In the light of belief polarization, we can think of $p\uparrow$ and $p\downarrow$ as positive and negative reactions to viewing some sort of documentation of p.

We define the updated model $\mathcal{M}^{p\uparrow} = (S, E, f^{p\uparrow}, || \bullet ||)$ where for all $e \in E$[1]:

$$f^{p\uparrow}(e) = \begin{cases} f(e) - 1 & \text{if } \exists s \in e \text{ such that } s \in ||p|| \text{ and } f(e) > 1, \\ f(e) & \text{otherwise.} \end{cases}$$

Similarly, we define the updated model $\mathcal{M}^{p\downarrow} = (S, E, f^{p\downarrow}, || \bullet ||)$ where for all $e \in E$:

$$f^{p\downarrow}(e) = \begin{cases} f(e) + 1 & \text{if } \exists s \in e \text{ such that } s \in ||p||, \\ f(e) & \text{otherwise.} \end{cases}$$

[1] It is straightforward to generalize this definition where we exchange 1 with a given natural number $\theta \in \mathbb{N}$. This is a standard approach in threshold models for social influence, seen in a modal logic context in work such as [5]. For now, we keep 1 as it represents the smallest possible change in f considering that its range is the set of natural numbers.

Returning to the example in Figure 0.2, we note that $P = \{w\}$, $Q = \{w, v\}$, and $R = \{t\}$. It follows that, for instance, for all $s \in S$, $s \in BQ$, but $s \notin BP$, since $Max_{\leq_E} S = \{w, v\} \subseteq Q$, but $Max_{\leq_E} S \not\subseteq P$. In other words, the agent believes Q, but not P. Now, consider the update $p \uparrow$. In the updated model $\mathcal{M}^{p\uparrow}$, f' is updated such that $f'(e_3) = f(e_3) - 1 = 1$, else $f'(e) = f(e)$. In turn, $f'(E_w) = 1$ while $f'(E_v) = f(E_v) = 2$, and thus this gives us a new updated plausibility order $w <'_E v <'_E t <'_E s$. It follows that $Max_{\leq'_E} S = \{w\}$ and hence in $\mathcal{M}^{p\uparrow}$, forall $s \in S : s \in BP$. In other words, after the update $p \uparrow$, the agent now believes p.

This update is quantitative in nature, as it leans on the definition of weighting models. To define a similar qualitative update, we exchange the function f in the weighting model with an ordering on the set E of evidence and then define an update that changes the ordering. A candidate for such a model is the justification model, also defined in [1], with appropriate refinements such that the update operations preserve the frame conditions. In on-going work, we provide a dynamic language with epistemic and doxastic operators, as well as updates $[p \uparrow]\phi$ and $[p \downarrow]\phi$, in order to obtain a sound and complete axiomatic system with respect to the class of weighting models.

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Sun, 12:10

Truthmaker Semantics and Hempelian Explanation in Mathematics

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Recent work on truthmaker semantics (TS) has extended the framework to provide truth conditions for counterfactuals [5], explain modality [6], and much more [4, 11, 12]. In this presentation, I argue for a Hempelian account of mathematical explanation based on a truthmaker-semantic framework capable of distinguishing between the states wholly relevant to a certain countermathematical and the inexact or loose mathematical consequences that would otherwise generate inconsistency.

A number of authors have argued that the natural sciences employ a class of distinctively mathematical explanations (DMEs) [10, 13, 3, 1]. Some accounts of DMEs explicitly employ Hempel's deductive-nomological model [7, 1, 10]. Lange, for example, contends that what makes DMEs distinctive is that they explain physical facts by constraint, that is, by appealing to a stronger modal force that renders any outcome other than the one described by the mathematical apparatus impossible. Why can't Ann divide the 23 apples equally among her 4 children? The reason is that 23 is not divisible by 4, and this mathematical fact explains why such a distribution is impossible. All physical outcomes must conform to this constraint, insofar as mathematical necessity is stricter than physical necessity. Viewed in this light, DMEs are, like explanations under Hempel's influential theory, arguments that use as their explanans a set of initial conditions and a number of mathematical facts or "laws" and that have as their conclusion the explanandum.

Some authors have advocated for understanding DMEs in terms of counterpossibles, in particular countermathematicals [3]. Counterpossibles are counterfactuals with impossible antecedents, and countermathematicals have mathematically impossible antecedents. As such, they are a special case of counterpossibles. If successful, this framing would align DMEs *qua* Hempelian explanations with established approaches in the philosophy of science, most notably Woodward's [14] interventionist account, by extending the framework to account for these kinds of specifically non-causal interventions.

For example, Baron, Colyvan, and Ripley [3] propose a procedure for evaluating countermathematicals such as: "If 13 were not prime, then cicadas would be eaten by their predators". On this approach, we "intervene" to hold fixed as much mathematical structure as possible while varying the specific feature that 13 is divisible only by 1 and itself, instead supposing it to be divisible by 2 and 6. Under this controlled deviation, the countermathematical is taken to be non-vacuously true.

Some have argued that Baron et al.'s account risks either collapsing into inconsistency (due to the propagation of incompatible mathematical properties once the antecedent is made true) or failing to be genuinely explanatory, insofar as it does not permit a meaningful space of variation comparable to that found in counterfactuals about physical systems [8].

In response to these concerns, Povich [13] draws on Kocurek and Jerzak's [9] account of counterpossibles about logic. If the previous account revived a Hempelian

view of explanation, this one follows in the tradition of Carnapian linguistic frameworks. It thus represents a neo-Carnapian approach. On this view, a counter-mathematical such as "Were 23 divisible by four, then Ann would have been able to divide the apples equally among her children" is to be understood as: "Were the semantic or conceptual rules governing the application of the term '23' such that, wherever it applied, 'divisible by four' also applied, Ann would have been able to divide the apples equally among her children". This reinterpretation, Povich argues, avoids the difficulties facing Baron et al.'s account. One advantage is that it offers a unified treatment of explanation in the natural sciences and mathematics as an extension of a conventionalist version of Woodward's interventionist framework. Nonetheless, concerns remain, particularly regarding the explanatory power of this approach (we are trying to explain the fact that Ann cannot divide the apples equally among her four children, not the fact that our semantic rules license that proposition), the subject matter of the paraphrased counterfactual (the counterfactual should be about what would have happened, had 23 been divisible by 4, not about semantic rules), and its commitment to a deflationary view of mathematics.

To address these objections, I make use of Fine's [5] modified account of exact truthmakers for counterfactuals. On this view, a state **s** is an exact verifier for $A > B$ iff it transforms exact verifiers of A into exact verifiers of B. In Fine's terms, **s** is "a fusion of paths" [5] from A's truthmakers to B's truthmakers. Taking the state of *13 not being prime* (**t**) as the unique exact verifier of "13 is not prime" (a simplification the atomicity of the antecedent licenses), what is required to establish the truth of "If 13 were not prime, then cicadas would be eaten by their predators" is an actual obtaining **s** such that $t \rightarrow_s u$, where **u** contains an exact verifier of "Cicadas are eaten by their predators". I argue that this **s** is a fusion of exactly those states we are justified in holding fixed under the supposition of **t** obtaining, with admissibility governed by the subject matter of the explanandum. The reasoning thus recovers the same inferential steps Povich identifies, but locates them in the mathematical states themselves rather than in semantic rules governing our terms. In this way we preserve the intuitive subject matter of the counterpossible while vindicating Baron et al.'s account: intervention becomes the stipulation of an impossible state as obtaining, and variation the consideration of the various outcome-states reachable by way of **s**.

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The Quest for Ultimate Ignorance I: Iterated Ignorance is Complex!

Fri, 10:25

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Ignorance is usually defined as the lack of knowledge. The concept of ignorance naturally leads to the ignorance operator $\mathcal{I}$ [3], where $\mathcal{I}\phi$ can be defined as not knowing whether ϕ. In terms of the knowledge operator $\mathcal{K}$, $\mathcal{I}$ can be defined as $\mathcal{I}\phi := \neg\mathcal{K}\phi \wedge \neg\mathcal{K}\neg\phi$.[1]

One can be ignorant not only about facts or about one's knowledge of them, but also about one's own ignorance, expressed by $\mathcal{I}^2\phi = \mathcal{I}\mathcal{I}\phi$ and further iterated any finite number of times $\mathcal{I}^n\phi = \mathcal{I}\ldots(n \text{ times})\ldots\mathcal{I}\phi$, as studied in [1]. This can be pushed further to the *ω-limit ignorance operator* $\mathcal{I}^\omega\phi$, where $\mathcal{I}^\omega\phi$ stands for the infinite conjunction $\bigwedge_{0<k<\omega}\mathcal{I}^k\phi$. Of course, the latter is not a syntactic definition but describes and gives an intuition of the semantics of $\mathcal{I}^\omega$.

Depending on the underlying logic of knowledge L, the iterations of $\mathcal{I}$ behave differently. In general, as we show, iteration of $\mathcal{I}$ need not stabilize (up to logical equivalence) with $\mathcal{I}^\omega$, or ever. Thus, for many background logics $\mathcal{I}^\omega$ is not yet the ultimate ignorance, so the quest for it must be extended transfinitely to $\mathcal{I}^\alpha$ for any ordinal α (in the naturally expected way) and it may never reach a stage where $\mathcal{I}^\alpha p \leftrightarrow \mathcal{I}^{\alpha+1}p$. If, however, that stabilisation ever occurs over a given underlying logic of knowledge L, for the least ordinal $\mathcal{I}^\alpha$ where that happens, we say that $\mathcal{I}^\alpha$ expresses *ultimate ignorance over* L and denote it by $\mathcal{I}^*_L$.

In this work, we explore the logical behaviour of the hierarchy of ignorance operators $\mathcal{I}, \mathcal{I}^2, \ldots, \mathcal{I}^\omega$ and leave the transfinite iterations of $\mathcal{I}$ to a follow-up work. We demonstrate that, contrary to common expectation, the behaviour of the hierarchy of iterated ignorance is generally rather complex, both in terms of the valid consequences between these operators and in terms of the semantic conditions corresponding to the respective implications between them above relatively weaker underlying logics of knowledge.

In particular, we show, inter alia, that:

- Many simple principles involving $\mathcal{I}$ and $\mathcal{K}$, such as $\mathcal{I}p \to \mathcal{I}\mathcal{I}p$, are not first-order definable, nor even over reflexive frames. Others, such as $\mathcal{I}\mathcal{I}p \to \mathcal{I}p$, are first-order definable, even canonical, but have non-trivial FO conditions.
- For each $k \in \mathbb{N}$ there is a finite reflexive model $\mathfrak{M}_k$ and a state w in it such that $\mathfrak{M}_k, w \vDash \mathcal{I}^i p$ iff $i \in [1, k]$. This strengthens a result by Fine [1].
- The knowledge of ω-ignorance is not satisfiable in reflexive frames. Formally, on reflexive frames, $\neg\mathcal{K}\mathcal{I}^\omega\phi$ is valid.

[1]Depending on the interpretation of the underlying modality $\Box$, the operator $\mathcal{I}$ generically defined as $\mathcal{I}\phi := \neg\Box\phi \wedge \neg\Box\neg\phi$ can also be interpreted alethically as *contingency* [2], or doxastically as *(belief) indifference*, or as *deductive independence* in the context of provability logics. Here, we only focus on its epistemic interpretation as ignorance.

- On the other hand, $\mathcal{I}^\omega p$ is satisfiable in a simple finite reflexive model.
- On all reflexive frames, $\mathcal{I}^\omega\phi \to \mathcal{I}\mathcal{I}^\omega\phi$ is valid, whereas $\mathcal{I}\mathcal{I}^\omega\phi \to \mathcal{I}^\omega\phi$ is not valid. Hence $\mathcal{I}^\omega\phi$ is not equivalent to $\mathcal{I}\mathcal{I}^\omega\phi$ on the class of reflexive frames.
- Moreover, for every $n > 1$, $\mathcal{I}^\omega p \to \mathcal{I}^n\mathcal{I}^\omega p$ is not valid on the class of reflexive frames, hence $\mathcal{I}^\omega\phi$ is not equivalent to any $\mathcal{I}^n\mathcal{I}^\omega\phi$. Thus, the hierarchy of iterated ignorance does not stabilise over the logic **T** even beyond ω.
- On the class of transitive frames, $\mathcal{I}^\omega\phi \to \mathcal{I}\mathcal{I}^\omega\phi$ is not valid either, though it is valid on all finite transitive frames. Consequently, the logic $\mathcal{L}(\mathcal{I}, \mathcal{I}^\omega)$ involving $\mathcal{I}$ and $\mathcal{I}^\omega$ as the only primitive operators does not have the finite model property over the class of transitive frames.
- On the other hand, as Fine showed in [1], over **S4** that hierarchy stabilises on level 2: for all $m, n \geq 2$, $\mathcal{I}^m\varphi \leftrightarrow \mathcal{I}^n\varphi$ is valid on all reflexive and transitive frames. We moreover show that over such frames, $\mathcal{I}^\omega\phi \leftrightarrow \mathcal{I}\mathcal{I}\phi$ is valid, hence $\mathcal{I}^2\phi$ is the ultimate ignorance over **S4**.

We also conjecture a simple, but infinitary axiomatisation over **K** for the logic $\mathcal{L}(\mathcal{I}, \mathcal{I}^\omega)$ comprising the following axiom and rule schemes:

1. $\mathcal{I}^\omega\phi \leftrightarrow \mathcal{I}^\omega\neg\phi$.
2. $\mathcal{I}^\omega\phi \to \mathcal{I}^n\phi$ for $n \geq 1$.
3. $\mathcal{I}^\omega\phi \leftrightarrow \mathcal{I}\phi \wedge \mathcal{I}^\omega\mathcal{I}\phi$.
4. $\psi \to \mathcal{I}^n\varphi$ for each $n \geq 1$ implies $\psi \to \mathcal{I}^\omega\varphi$.

The completeness of the above axiomatic system is currently still open. The questions whether $\mathcal{L}(\mathcal{I}, \mathcal{I}^\omega)$ has a finitary axiomatization over the class of all frames, or over any of the subclasses mentioned above where the hierarchy of iterated ignorance does not stabilise on a finite level, are currently open, too. These and other currently open related technical questions, about finite model property and decidability, will be explored in a follow-up to the present work.

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Formalizing a Concise Axiomatic System for Classical Propositional Logic in Isabelle/HOL

Sun, 15:45

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Technical University of Denmark

Axiomatic systems serve as the bedrock for deductive logic. We focus on the soundness and completeness theorems for a well-known system by Jan Łukasiewicz (his main axiomatic system in the book *Elements of Mathematical Logic*, Pergamon Press 1963; Polish First Edition 1929 & Second Edition 1958) and formalize it in Isabelle/HOL [5] (the default higher-order logic of the Isabelle proof assistant).

A recent survey [4, page 4] on tools for teaching the metatheory of logic states:

> *T*eaching logic can rely on relatively low additional machinery as proof assistants expose natively their underlying logic. Exercising natural deduction, sequent-calculus is thus relatively straightforward, and numerous experiments have been conducted using general purpose proof assistant or ad hoc software. Using proof assistants to teach the meta-theory of logic is rarer. Some authors such as Villadsen and his co-authors have proposed using proof assistants with a deep embedding of some calculus, allowing formalization of meta-theory to teach both logic and automated deduction [2, 6, 7].

Metatheory is important for a full understanding of logic [1, 3]. Most logic textbooks cover the soundness and completeness theorems of propositional logic. We consider the *modus ponens* rule and the following three axioms (implication associates to the right as in Isabelle/HOL). Axiom 1 $(q \to r) \to (r \to p) \to q \to p$ represents the *transitivity of implication*, allowing direct chaining of logical consequences. Axiom 2 $(\neg p \to p) \to p$ is *Clavius's Law*, enabling classical reasoning by deducing a proposition if its negation implies itself. Axiom 3 $q \to \neg q \to p$ is the *principle of explosion*, asserting that any arbitrary proposition follows from a contradiction.

The formalization uses a *deep embedding*, where the logic is defined as a datatype within Isabelle/HOL. This setup allows users to prove theorems *about* the logic rather than just deriving results *within* it.

Our goal is to start with propositional logic and then use the same machinery, that is, Isabelle definitions, theorems, and proofs, for first-order logic. Often propositional logic uses, for the completeness proof, a separate machinery (essentially based on truth tables and the like) that cannot be used for first-order logic (with a domain of quantification). Our main contribution is a formalization of propositional logic extensible to first-order logic, contributing 350 lines to a recent 1800-line formalization of first-order logic for languages of arbitrary cardinalities developed by Villadsen et al.[*]

In summary, our approach is to formalize and verify everything in Isabelle, such that students can examine and modify the formal proofs for the metatheory, for

[*] Isabelle Formalization of Logic: https://github.com/IsaFoL/IsaFoL/tree/master/Axiomatics

example to change the choice of primitive logical operators as well as the axioms and rules for the axiomatic systems. We have developed and systematically evaluated exercises, assignments, and exam problems for this. This setup provides a streamlined path for students and researchers to learn about the soundness and completeness theorems.

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The Definability of Truth in LSO

Fri, 13:45

Michal Walicki
University of Bergen

Logic of sentential operators, LSO introduced in [10], extends classical logic with sentential operators and quantifiers, allowing to address statements without coding/naming them. If we read the operator $K\phi$ as 'Cretan K saying sentence ϕ', then $\forall\phi(K\phi \rightarrow \neg\phi)$ says 'everything Cretan K says is false', while $K(\forall\phi(K\phi \rightarrow \neg\phi))$ says that K is saying that. The metalanguage, using the sentential quantifiers and operators, is thus part of the language. Paradoxes in LSO are such metalinguistic claims arising from the unfortunate valuations of operators that fail to extend to the entire language. Their expressibility requires no artificial restrictions to maintain consistency—just the usual care in formulating one's claims. The liar paradox arises, e.g., when K says *only* $\forall\phi(K\phi \rightarrow \neg\phi)$, but not if K says also something true.

Any classical language $\mathcal{L}^{-}$ (we use FOL as the natural example) can be expanded to $\mathcal{L}^{\doteq}$ with the sentential quantifiers and operator $\doteq$ holding for the syntactically identical sentences. We consider any expansion $\mathcal{L}$ of $\mathcal{L}^{\doteq}$ with arbitrary operators. The interpretation of $\mathcal{L}$ uses a digraph $G_M(\mathcal{L})$, for each FOL domain M. The notion of a digraph kernel generalises the well-founded semantics of classical logic to the graph cycles which capture self-referential statements. The semantics is equivalent to the classical one when restricted to $\mathcal{L}^{-}$. LSO is classical also in the sense of being two-valued and retaining all rules of the classical sequent system, augmenting it with two rules for the sentential quantifiers. These are FOL quantifier rules for the sort of sentences, so the reasoning is complete and the logic remains compact. (Modal logic with Kripke semantics is not addressed, but its formulation as a specialisation of LSO might deserve a closer study).

Building on this earlier treatment of self-reference and paradoxes, we formalise the truth of LSO sentences in LSO. The replacement of predicates by operators circumvents the fundamental dilemma requiring to renounce either the classical logic or the definability of truth. Convention T – formulated as (T) $\forall\phi(T\phi \leftrightarrow \phi)$ – acts in the way proposed by Tarski: as the necessary condition of material adequacy for a definition of truth. In the operator setting, it can be taken as a definition introducing a trivial, identity-like operator. A satisfactory definition, however, should possess some substance, even if it is unclear what exactly this means. One would naturally expect it to arise from the language's ability to capture its own semantics and to reflect how it makes some sentences true and others false. We axiomatise the semantic structure in LSO and provide a definition of truth that implies (T), but is not implied by it. No sufficient criteria for truth are considered and any consistent theory that implies (T) might qualify as a candidate whose merits must be assessed on broader grounds.

Expanding any language $\mathcal{L}$ to $\mathcal{L}^{T}$ with the operators for the edge relation and truth, we axiomatise first the graph interpretation of the language, by schemas over $\mathcal{L}^{T}$ formulas. The axiomatisation of truth of $\mathcal{L}^{T}$ sentences reflects then the definition of a digraph kernel. It implies (T), captures how the graph structure makes sentences true or false, and is model conservative. Not relying on coding

of sentences, it requires no syntax theory and can augment any consistent formalisation of syntax. The uniform formulation – for $G_M(\mathcal{L}^T)$ over all domains M – captures validity of $\mathcal{L}^T$ sentences. Truth in a specific structure can be defined as an instance of this general notion when the structure is axiomatisable.

In response to a reviewer's request to "expand the abstract with more context and related work", there is little else to do but to comment on the differences from the main other approaches. The extensive literature on truth-theories, following Tarski's seminal paper [9], forms part of the broader context but is not directly related, since it investigates truth predicates and not operators. Likewise the revision operators, in revision theories following [2], are meta-theoretic devices used to define truth predicates. Truth operator (or "connective") occurs less formally in linguistic or philosophical discussions, e.g., [4, 8], but we are not aware of any formalism related to LSO, if by "related" we mean sufficiently close technical solutions suggesting potential interactions. One might expect some similarities to the modal approaches, but LSO does not use Kripke semantics. The truth operator considered in [6] is formulated within a non-classical logic, whereas classical modal approaches to truth typically rely on Gödelisation and hence remain within a predicate-based framework (as do provability logics). For instance, modal logics developed in [5, 7] aim to recreate the predicates of various truth-theories; the truth operator in [11], defined trivially, acts along a truth predicate. As soon as one quantifies over the arguments of a truth operator, there emerges the need for a predicate: *"since 'for all x ... it is true that x ...' is not well-formed in standard first-order languages, how shall we relax our syntactic regimentation in order to harmonize truth operators with quantification? Using a truth predicate avoids all these complications from the start"*, [3]. Moreover, *"the problems of interpreting propositional [sentential] quantifiers and the truth predicate are exactly parallel. They suffer from the same sort of impredicativity"*, [1]. The handling of this impredicativity is a distinctive feature of LSO which, together with the use of an operator rather than a predicate, underlies the consistency of the proposed definition.

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Licensing Hypothetical Reasoning

Uwe Wolter
University of Bergen

Logics of Statements in Context (LoSiCs) have been introduced in [3] as a general framework enabling us to describe, analyze and relate a variety of diverse concepts, results and constructions from a broad spectrum of logics and specification formalisms including First-Order Logic, Universal Algebra, Algebraic Specifications, Ehresmann Sketches, Generalized Sketches [1, 2], Description Logic, Graph Transformations and Software Modeling, for example.

The crucial idea behind the concept of *context* is to establish an additional conceptual layer between syntax (signatures, constant symbols, variables, terms, formulas) and semantics (structures and their carriers).

When it comes to *open formulas*, we are faced with the situation that deduction in traditional FOL is, in fact, deduction of *open formulas* while the meaning of *open formulas* cannot be described in terms of first-order structures as such, in contrast to *closed formulas*! Nevertheless, the deduction calculi turn out to be sound and complete for *closed formulas*. LoSiCs propose a uniform way to define logics also comprising *open formulas* and to potentially resolve this somehow paradoxical phenomenon in a precise formal way. As a first step, it has been shown in [3] that we can define arbitrary first-order *open formulas* (and corresponding *statements in context*), and their semantics in arbitrary categories and not only in the category Set.[1]

Treating open formulas as first-class citizens, goes hand-in-hand with a shift of perspective: The LoSiC-approach relies on the insight that the difference between closed formulas and open formulas boils down to the difference between empty contexts and non-empty contexts. *Statements in context* are statements about single (!) interpretations of contexts in a Σ-structure $\mathcal{U}$. For the empty context there is, however, only exactly one interpretation in a Σ-structure $\mathcal{U}$, thus a statement in the empty contexts becomes a statement about a Σ-structure $\mathcal{U}$ as such!

[4] presents an analysis of three deduction calculi for traditional unsorted FOL in view of LoSiCs. Instead of relying on sets of free variables syntactically appearing in open formulas, we propose to work with explicit declarations of *sets of available free variables* as *contexts*. Let Σ be an arbitrary first-order signature. We call a pair (P, Γ) of a set P of variables and a set Γ of well-formed first-order Σ-formulas, with free variables only from P, a *Σ-sketch*. Each $A \in \Gamma$ is implicitly bound to the context P by the inclusion map $in_A : fv(A) \hookrightarrow P$ and considered as a *statement in context* P. The semantics $[\![(P, \Gamma)]\!]^{\mathcal{U}} \subseteq U^P$ of (P, Γ) in a Σ-structure $\mathcal{U}$ is given by all variable assignments $\mathbf{a} : P \to U$ making all the formulas in Γ true.

Adapting [2], let a *Σ-sketch implication* $(P, \Gamma) \Rightarrow (K, \Delta)$ to be given by Σ-sketches (P, Γ), (K, Δ) such that $P \subseteq K$. $(P, \Gamma) \Rightarrow (K, \Delta)$ is *satisfied* in a Σ-structure $\mathcal{U}$, $\mathcal{U} \models^{\Sigma} (P, \Gamma) \Rightarrow (K, \Delta)$ in symbols, if each $\mathbf{a} \in [\![(P, \Gamma)]\!]^{\mathcal{U}} \subseteq U^P$ can be extended to a $\mathbf{b} \in [\![(K, \Delta)]\!]^{\mathcal{U}} \subseteq U^K$.

[1] LoSiCs don't qualify as *Categorical Logic* even if we use some basic categorical concepts and constructions to develop and present them! If a label is required, we would rather label them as *Abstract Model Theory*.

Σ-sketch implications $(P,\Gamma) \Rightarrow (K,\Delta)$ can also serve as deduction rules to derive, in a sound way, other Σ-sketch implications $(G,\Phi) \Rightarrow (R,\Psi)$ applying them to sketches (G,Φ) via *matches* $\mathbf{t} : (P,\Pi) \dashrightarrow (G,\Phi)$ given by substitutions $\mathbf{t} : P \to T_\Sigma(G)$ translating Π into a subset of Φ. We show in [4] that this mechanism is sufficient to describe and validate all the rules in a Hilbert System $\mathcal{H}$, a Gentzen's system $\mathcal{LJ}$ and a Natural Deduction System $\mathcal{ND}$, except the rules relying on *hypothetical reasoning*. In the talk we will focus on System $\mathcal{ND}$.

We introduce *open Σ-sketch implications* $I \to (P,\Pi) \Rightarrow (K,\Delta)$, given by two Σ-sketches (P,Π), (K,Δ) with $P \subseteq K$ and an "interface" $I \subseteq P$, to formally grasp hypothetical reasoning. The semantics $[\![I \to (P,\Pi) \Rightarrow (K,\Delta)]\!]^{\mathcal{U}} \subseteq U^I$ of $I \to (P,\Pi) \Rightarrow (K,\Delta)$ in a Σ-structure $\mathcal{U}$ is given by those $\mathbf{a} \in U^I$ such that for all extensions $\mathbf{b} : P \to U$ of $\mathbf{a} : I \to U$ along $i : I \hookrightarrow P$ satisfying Π there exists an extension $\mathbf{c} : K \to U$ along $in : P \hookrightarrow K$ satisfying Δ.

In the talk we outline that open Σ-sketch implications allow us to describe the results of hypothetical proofs "$[A] \cdots B$" in a precise formal way and independent of the concrete proofs! This is based on the observation that introducing a hypothesis "$[A]$" relies on the construction of a pushout of inclusion maps. Especially, one can observe that *Skolem constants* are nothing but simply a vehicle to construct those pushouts! In the talk we will discuss that open Σ-sketch implications can also be utilized as *locally bound deduction rules*! Let us have a look at rule $\forall I$:

$$\mathcal{LJ}\colon \forall I\ \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ x \notin fv(\Gamma) \qquad \mathcal{ND}\colon\ \forall I^*\ \frac{A}{\forall x A} \qquad \begin{array}{c}[c]^1 \\ \vdots \\ A_c^x \\ \hline \forall x A\end{array}\ \forall I^1 \quad \text{Intuition:} \tag{1}$$

* means that A does not depend on any hypothesis in which x is free.

The informal version *Intuition* indicates how we could recast, in the spirit of hypothetical reasoning, the puzzling condition "A does not depend on any hypothesis in which x is free" by introducing a "new Skolem constant c". As any rule in $\mathcal{LJ}$, also rule $\forall I$ is an implication between Σ-sketch implications! It states that $\mathcal{U} \models (Z,\Gamma) \Longrightarrow (Z,A)$ implies $\mathcal{U} \models (Y,\Gamma) \Longrightarrow (Y,\forall x A)$ for any Σ-structure $\mathcal{U}$, where $Z = fv(\Gamma) \cup fv(A) \cup \{x\}$ and $Y = Z \setminus \{x\}$. The open Σ-sketch implication $I \to (P,\emptyset) \Rightarrow (P,A)$ with $P = fv(A) \cup \{x\}$, $I = P \setminus \{x\}$ turns out to be the crucial ingredient to validate this rule as well in system $\mathcal{LJ}$ as in system $\mathcal{ND}$.

In case of system $\mathcal{LJ}$, one can show, by the soundness of hypothetical reasoning, that for any $\mathbf{a} \in [\![(Y,\Gamma)]\!]^{\mathcal{U}} \subseteq U^Y$ the restriction $j;\mathbf{a} : I \to U$ of $\mathbf{a} : Y \to U$ along the inclusion map $j : I \hookrightarrow Y$ is an element in $[\![I \to (P,\emptyset) \Rightarrow (P,A)]\!]^{\mathcal{U}}$ as long as $\mathcal{U} \models (Z,\Gamma) \Longrightarrow (Z,A)$. This means, in other words, that all extensions $\mathbf{b} \in U^Z$ of $\mathbf{a} \in [\![(Y,\Gamma)]\!]^{\mathcal{U}}$ satisfy A! In case of system $\mathcal{ND}$ the soundness of rule $\forall I^*$ can analogously be shown by replacing the pair (Z,Γ), with Z only containing variables and Γ only containing Σ-formulas, by a pair (G,Φ) with G not only containing variables but also Skolem constants and with Φ containing not only open formulas but also open specification implications.

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A Jónsson-Tarski-Goldblatt Representation and Frame Definability for Noncontingency Logic

Sat, 12:05

Qianli Zeng
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In standard normal modal logic, the Jónsson-Tarski duality and the subsequent Goldblatt-Thomason theorem rely fundamentally on the normality and additivity of the modal operator $\Box$. The logic of noncontingency, denoted $\mathcal{L}(\Delta)$, introduces an operator Δ that lacks both normality and additivity.

Intuitively, a proposition is contingent if it is possibly true and possibly false; it is non-contingent if it is necessarily true or necessarily false. In an epistemic setting, the notion of contingency corresponds to different logical contexts: it represents 'ignorance' in epistemic logic and 'no belief' in doxastic logic, while noncontingency can be interpreted as the natural statement of 'knowing whether φ' [2].

The language $\mathcal{L}$ of noncontingency logic K_Δ is defined as follows:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Delta\varphi$$

Given a Kripke frame with a set of worlds W and relation R, the truth condition for the noncontingency operator is: $s \models \Delta\varphi$ iff either $\forall t(sRt \Rightarrow t \models \varphi)$ or $\forall t(sRt \Rightarrow t \not\models \varphi)$.

Because the standard Jónsson-Tarski duality does not apply directly to this non-normal operator, we provide a custom representation for noncontingency algebras and a Goldblatt-Thomason frame definability characterization theorem.

We first define the algebraic counterpart to the minimal contingency logic K_Δ.

Definition 1 (Noncontingency Algebra (NCA)). *A structure $\mathfrak{A} = (A, +, -, 0, 1, \Delta)$ is a Noncontingency Algebra (NCA) if $(A, +, -, 0, 1)$ is a Boolean algebra, and the unary operator $\Delta : A \to A$ satisfies the following identities for all $x, y \in A$:*

1. $\Delta 1 = 1$
2. $\Delta x = \Delta(-x)$
3. $\Delta x \leq \Delta(-x + y) + \Delta(x + y)$
4. $\Delta(-y + x) \cdot \Delta(y + x) \leq \Delta x$

Because Δ does not distribute over meets, $\{x \in A \mid \Delta x \in u\}$ does not form a filter. To construct the representation, we require a contextual necessity operator $L_z(x) := \Delta x \cdot \Delta(-z + x)$. This algebraic construct corresponds to the "almost-definability" schema (AD) proposed by Fan, Wang and van Ditmarsch [2]. The AD schema demonstrates how necessity can be defined using noncontingency under specific preconditions, expressed as:

$$\neg\Delta\psi \to (\Box\varphi \leftrightarrow (\Delta\varphi \wedge \Delta(\psi \to \varphi)))$$

This approach offers a uniform method to show completeness over various frame classes, avoiding the scattered and non-uniform methods previously used in the literature [3]. For any ultrafilter $u \in Uf(\mathfrak{A})$ and contingent witness $z \in A$ (where

$-\Delta z \in u$), the set of z-necessitated elements $\lambda(u,z) := \{y \in A \mid L_z(y) \in u\}$ uniquely forms a proper filter on A.

Definition 2 (Canonical Ultrafilter Frame). *For an NCA $\mathfrak{A}$, the canonical ultrafilter frame is $\mathfrak{A}_+ = (Uf(\mathfrak{A}), R_\Delta)$, where $uR_\Delta v$ iff:*

1. If $\forall x \in A, \Delta x \in u$, then $u = v$.
2. If $\exists x \in A$ such that $-\Delta x \in u$, then for all $y \in A$, if $L_x(y) \in u$, then $y \in v$.

This allows us to establish the noncontingency analog of the Jónsson-Tarski embedding.

Theorem 1 (Representation Lemma). *Let $r : A \to \mathcal{P}(Uf(\mathfrak{A}))$ be defined by $r(x) = \{u \mid x \in u\}$. Then for any $x \in A$, $r(\Delta x) = \Delta_{R_\Delta}(r(x))$, where $\Delta_{R_\Delta}(Y) = \{u \mid R_\Delta[u] \subseteq Y$ or $R_\Delta[u] \subseteq Uf(\mathfrak{A}) \setminus Y\}$.*

Consequently, the map $r : \mathfrak{A} \hookrightarrow Cm(\mathfrak{A}_+)$ constitutes an embedding, establishing every NCA is isomorphic to a subalgebra of the complex noncontingency algebra of its canonical ultrafilter frame.

To establish the definability theorem, we translate the algebraic closure operations into their duals over the frame class. The following operations are due to Jie Fan.

Definition 3 (Δ-Generated Subframe). *A subset $V \subset W$ is a Δ-generated subframe if and only if, for all $v \in V$, one of the following holds:*

1. $R_W[v] \subseteq V$.
2. $R_W[v] \cap V = \emptyset$ and $|R_W[v]| = 1$.

Definition 4 (Δ-Bounded Morphism). *A map $f : W \to W'$ is a Δ-bounded morphism if and only if, for all $w \in W$, one of the following holds:*

1. $f[R[w]] = R'[f(w)]$.
2. $|f[R[w]]| \leq 1$ and $|R'[f(w)]| \leq 1$.

We demonstrate that surjective NCA homomorphisms dualize to injective Δ-bounded morphisms, and that algebraic subalgebras correspond to bounded morphic images. Combining these results via Birkhoff's HSP theorem yields the characterization.

Theorem 2 (Goldblatt-Thomason for $\mathcal{L}(\Delta)$). *Let $\mathbb{K}$ be a first-order definable class of Kripke frames. $\mathbb{K}$ is definable in $\mathcal{L}(\Delta)$ if and only if $\mathbb{K}$ is closed under disjoint unions ($\biguplus$), Δ-generated subframes ($\hookrightarrow_\Delta$), Δ-bounded morphic images ($\twoheadrightarrow_\Delta$), and reflects Δ-ultrafilter extensions (ue_Δ).*

As a corollary, any non-empty Δ-definable frame class $\mathbb{K}$ must contain the class of all partial functional frames $\mathbb{F}_{func}$ [4]. Therefore, standard relational properties like reflexivity, symmetry, and transitivity are not definable in the $\mathcal{L}(\Delta)$ language.

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